

A MARTINGALE PROBLEM FOR AN ABSORBED DIFFUSION: THE NUCLEATION PHASE OF CONDENSING ZERO RANGE PROCESSES

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ABSTRACT. We prove uniqueness of a martingale problem with boundary conditions on a simplex associated to a differential operator with an unbounded drift. We show that the solution of the martingale problem remains absorbed at the boundary once it attains it, and that, after hitting the boundary, it performs a diffusion on a lower dimensional simplex, similar to the original one. We also prove that in the diffusive time scale condensing zero-range processes evolve as this absorbed diffusion.

1. INTRODUCTION

It has been observed, in several different contexts, that some zero-range processes whose jump rates decrease to a positive constant exhibit condensation: in the stationary state, above a certain critical density, a macroscopic fraction of the particles concentrate on a single site [11, 9, 6, 8, 1, 2, 5, 3].

We investigated in [5, 13] the evolution of the condensate in the case where the total number of sites remains fixed while the total number of particles diverges. We proved that in an appropriate time scale the condensate – the site where all but a negligible fraction of particles sit – evolves according to a Markov chain whose jump rates are proportional to the capacities of the underlying random walks performed by the particles in the zero-range process.

We examine in this article how the condensate is formed in the case where the set of sites, denoted by S , remains fixed while the total number of particles, N , tends to infinity. Consider an initial configuration in which each site is occupied by a positive fraction of particles. Since in the stationary state almost all particles occupy the same site, as time evolves we expect to observe a progressive concentration of particles on a single site.

Absorbed diffusions. To describe the asymptotic dynamics we were led to analyze a diffusion with boundary conditions whose drift diverges as the process approaches the boundary and which remains glued to the boundary once it attains it. More precisely, denote by Σ the simplex $\{x \in \mathbb{R}^S : x_j \geq 0, \sum_j x_j = 1\}$, where x_j represents the fraction of particles at site $j \in S$. Far from the boundary of Σ the process evolves as a standard diffusion with bounded and smooth coefficients, whereas close to the boundary $\{x \in \Sigma : x_j = 0\}$ the drift becomes proportional to b/x_j , where $b > 1$ is a fixed parameter, while the variance remains bounded by 1. In consequence, when the process approaches the boundary of Σ it is strongly driven to it. Once the boundary $\{x \in \Sigma : \sum_{j \in A} x_j = 0\}$, $A \subset S$, is attained, the process remains absorbed at this boundary, where it performs a new diffusion, similar to the original one, but in a lower dimensional space. This mechanism is iterated and the dimension of the space in which the diffusion occurs decreases progressively until all but

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one coordinate vanish. At this point the process remains trapped in this configuration for ever.

We called these dynamics “absorbed” diffusions to distinguish them from “sticky” diffusions [10] which bounces at the boundary, but which have a positive local time at the boundary.

We did not find in the literature examples of diffusions whose absorption at the boundary arises from a divergence of the drift close to the boundary. Diffusions which are absorbed at the boundary due to a singularity of the covariance matrix of Wright-Fisher type have been examined in [7].

Hydrodynamic limit. The most popular methods to derive the hydrodynamic equations of the conserved quantity of an interacting particle systems relies on the so-called one and two block estimates [12]. The condensing zero-range processes examined in this article form a class of dynamics for which the one and two blocks estimate in their classical form do not hold, precisely because of the condensation of particles.

We examine in this article how particles accumulate on a single site in the diffusive time scale when the total number of sites is fixed and the total number of particles diverges. This is called the nucleation phase of the condensing zero-range dynamics. A most interesting open problem is the behavior of the same model, when the number of sites increases together with the number of particles, where a convergence towards a self-similar distribution is expected, or the proof of the hydrodynamical limit of the model when the initial density profile is super-critical. A first step in this direction has been performed in [14], where an alternative version of the one block estimate is proved together with the hydrodynamical limit for initial profiles bounded below by the critical density, a situation in which there is no condensation in the diffusive time scale.

Results. The first main result of this article asserts that there exists a unique solution to the martingale problem associated to a second-order differential operator of an absorbed diffusion. The second main result states that in the diffusive time scale N^2 the fraction of particles of condensing zero-range processes on finite sets evolves as an absorbed diffusion.

We faced two main obstacles in this article. The first consisted in the proof that the solution of the martingale problem remains absorbed at the boundary once it attains it, and that, after hitting the boundary, the solution performs a diffusion on a lower dimensional space, similar to the original one. These results and Stroock and Varadhan [15, 16] theory, with some slight modifications due to the unboundedness of the drift, yield uniqueness of the martingale problem. The second main difficulty consisted in the proof of the tightness of the condensing zero-range processes in the diffusive time scale, which required a replacement lemma.

2. NOTATION AND RESULTS

We present in this section the two main results of the article.

2.1. The underlying Markov chain. Fix a finite set $S = \{1, \dots, L\}$, and consider an irreducible, continuous-time Markov chain $(x_t)_{t \geq 0}$ on S . Denote by $\mathbf{r} = \{r(j, k) : j, k \in S\}$ the jump rates, so that the generator \mathcal{L} of this Markov chain is

$$(\mathcal{L}f)(j) = \sum_{k \in S} r(j, k) \{f(k) - f(j)\}.$$

Assume, without loss of generality, that $r(j, j) = 0$ for all $j \in S$, and denote by $\lambda(j)$ the holding rate at j , $\lambda(j) = \sum_{k \in S} r(j, k)$. Let $\mathbf{m} = \{m_j : j \in S\}$ be an invariant measure for \mathbf{r} , and let $M_j = m_j \lambda(j)$, $j \in S$, so that M_j is an invariant measure for the

embedded discrete-time chain. Note that we do not assume \mathbf{m} to be a probability measure nor reversible for \mathbf{r} .

2.2. Condensing zero-range processes. Denote by η, ξ the elements of \mathbb{N}^S , $\mathbb{N} := \{0, 1, 2, \dots\}$, so that $\eta(j)$, $j \in S$, represents the number of particles at site j for the configuration η . Denote by E_N , $N \geq 1$, the set of configurations with N particles:

$$E_N := \left\{ \eta \in \mathbb{N}^S : \sum_{j \in S} \eta(j) = N \right\}.$$

Fix $b > 1$. For each $j \in S$, consider a jump rate $g_j : \mathbb{N} \rightarrow \mathbb{R}_+$ such that $g_j(0) = 0$ and

$$\lim_{n \rightarrow \infty} n \left(\frac{g_j(n)}{m_j} - 1 \right) = b. \quad (2.1)$$

In particular, $g_j(n) \rightarrow m_j$ and there exists $n_0(j) \in \mathbb{N}$ such that $g_j(n) > m_j$ for all $n \geq n_0(j)$. Hence, roughly, each function g_j decreases and converges to m_j at rate (2.1).

Denote by $\eta^N = (\eta_t^N)_{t \geq 0}$ the zero range process on S associated to the jump rates $r(j, k)$ and g_j . This is the continuous-time Markov chain on \mathbb{N}^S in which a particle jumps from j to k at rate $g_j(\eta(j))r(j, k)$. The generator L_N of this chain acts on functions $F : E_N \rightarrow \mathbb{R}$ as

$$(L_N F)(\eta) = \sum_{j, k \in S} g_j(\eta(j)) r(j, k) [F(\sigma^{j, k} \eta) - F(\eta)], \quad (2.2)$$

where $\sigma^{j, k} \eta$ stands for the configuration obtained from η by moving a particle from j to k :

$$(\sigma^{j, k} \eta)(\ell) = \begin{cases} \eta(j) - 1, & \text{for } \ell = j \\ \eta(k) + 1, & \text{for } \ell = k \\ \eta(\ell), & \text{otherwise.} \end{cases}$$

Denote by Σ the simplex

$$\Sigma = \left\{ x \in \mathbb{R}_+^S : \sum_{j \in S} x_j = 1 \right\},$$

and by Σ_N the N -discretization of Σ :

$$\Sigma_N = \left\{ x \in \Sigma : Nx_j \in \mathbb{N}, j \in S \right\}, \quad N \geq 1.$$

Fix a configuration $\eta \in E_N$ and let η^N be started at η . We consider the rescaled process

$$X_t^N := \frac{1}{N} (\eta_{tN^2}^N(1), \dots, \eta_{tN^2}^N(L)), \quad t \geq 0.$$

Clearly, $(X_t^N)_{t \geq 0}$ is a Σ_N -valued Markov chain whose generator \mathfrak{L}_N acts on functions $F : \Sigma_N \rightarrow \mathbb{R}$ as

$$(\mathfrak{L}_N F)(x) = N^2 \sum_{j, k \in S} g_j(Nx_j) r(j, k) \left[F\left(x + \frac{e_k - e_j}{N}\right) - F(x) \right], \quad (2.3)$$

where $\{e_i : i \in S\}$ represents the canonical basis of \mathbb{R}^S .

2.3. Martingale problem. One of the main result of this article states that the Markov chain X_t^N converges in law to a diffusion on Σ . To introduce the generator of the diffusion we first define its domain. Denote by $C^n(\Sigma)$, $n \geq 1$ the set of functions $F : \Sigma \rightarrow \mathbb{R}$ which are n -times continuously differentiable. We let $\partial_{x_k} F$ and ∇F stand for partial derivative with respect to the variable x_k and for the gradient, respectively, of $F \in C^1(\Sigma)$.

Let $\{v_j \in \mathbb{R}^S : j \in S\}$ be the vectors

$$v_j := \sum_{k \in S} r(j, k) \{e_k - e_j\}, \quad j \in S,$$

and define the vector field $\mathbf{b} : \Sigma \rightarrow \mathbb{R}^S$ by

$$\mathbf{b}(x) := b \sum_{j \in S} \mathbf{1}\{x_j \neq 0\} \frac{m_j}{x_j} v_j. \quad (2.4)$$

Definition 2.1. For each $j \in S$, let \mathcal{D}_j , be the space of functions H in $C^2(\Sigma)$ for which the map $x \mapsto [v_j \cdot \nabla H(x)]/x_j \mathbf{1}\{x_j > 0\}$ is continuous on Σ , and let $\mathcal{D}_A := \cap_{j \in A} \mathcal{D}_j$, $\emptyset \subsetneq A \subseteq S$.

Clearly, if H belongs to \mathcal{D}_j , $v_j \cdot \nabla H(x) = 0$ for any $x \in \Sigma$ such that $x_j = 0$. Moreover, if H belongs to \mathcal{D}_S , $x \mapsto \mathbf{b}(x) \cdot \nabla H(x)$ is continuous on Σ . Finally, to prove that a function H in $C^2(\Sigma)$ belongs to \mathcal{D}_j , we need to show that for every $z \in \Sigma$ such that $z_j = 0$,

$$\lim_{\substack{x \rightarrow z \\ x_j > 0}} \frac{v_j \cdot \nabla H(x)}{x_j} = 0. \quad (2.5)$$

In this formula, the limit is carried over points x in Σ which converge to z and such that $x_j > 0$.

Let us now introduce the generator for the limiting diffusion. Let $\mathfrak{L} : C^2(\Sigma) \rightarrow \mathbb{R}$ be the second order differential operator given by

$$(\mathfrak{L}F)(x) := \mathbf{b}(x) \cdot \nabla F(x) + \frac{1}{2} \sum_{j, k \in S} m_j r(j, k) (\partial_{x_k} - \partial_{x_j})^2 F(x), \quad (2.6)$$

for any $F \in C^2(\Sigma)$. Thus, operator \mathfrak{L} depends on three parameters: the coefficient $b > 1$, the jump rates r and the measure \mathbf{m} . Of course, for $H \in \mathcal{D}_S$, the function $\mathfrak{L}H : \Sigma \rightarrow \mathbb{R}$ is continuous.

Let $\langle \cdot, \cdot \rangle_{\mathbf{m}}$ be the L^2 -inner product with respect to the measure \mathbf{m} . Denote by $\mathbf{a} = (a_{i,j})_{i,j \in S}$ the matrix whose entry (i, j) is given by $a_{i,j} = \langle e_i, -\mathcal{L}e_j \rangle_{\mathbf{m}}$ and denote by \mathbf{a}_s the symmetric matrix $\mathbf{a}_s = (1/2)(\mathbf{a} + \mathbf{a}^t)$ where \mathbf{a}^t stands for the transpose of \mathbf{a} . So \mathbf{a}_s is the matrix corresponding to the Dirichlet form of the symmetric part of $(\mathcal{L}, \mathbf{m})$. With this notation we may write the generator \mathfrak{L} as

$$(\mathfrak{L}F)(x) = \mathbf{b}(x) \cdot \nabla F(x) + \text{Tr}[\mathbf{a}_s \times \text{Hess } F(x)], \quad (2.7)$$

where $\text{Tr } \mathbb{M}$ stands for the trace of the matrix \mathbb{M} and $\text{Hess } F$ for the Hessian of F .

We now characterize the limiting diffusion as the solution of the martingale problem corresponding to $(\mathfrak{L}, \mathcal{D}_S)$. Denote by $C(\mathbb{R}_+, \Sigma)$ the space of continuous trajectories $\omega : \mathbb{R}_+ \rightarrow \Sigma$ endowed with the topology of uniform convergence on bounded intervals. Every probability measure on $C(\mathbb{R}_+, \Sigma)$ will be defined on the corresponding Borel σ -field \mathcal{F} . We denote by $X_t : C(\mathbb{R}_+, \Sigma) \rightarrow \Sigma$, $t \geq 0$ the process of coordinate maps and by $(\mathcal{F}_t)_{t \geq 0}$ the generated filtration $\mathcal{F}_t := \sigma(X_s : s \leq t)$, $t \geq 0$. A probability measure \mathbb{P} on $C(\mathbb{R}_+, \Sigma)$ is said to start at $x \in \Sigma$ when $\mathbb{P}[X_0 = x] = 1$. In addition, we shall say that \mathbb{P} is a solution for the \mathfrak{L} -martingale problem if, for any $H \in \mathcal{D}_S$,

$$H(X_t) - \int_0^t (\mathfrak{L}H)(X_s) ds, \quad t \geq 0 \quad (2.8)$$

is a \mathbb{P} -martingale with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

Theorem 2.2. *For each $x \in \Sigma$, there exists a unique probability measure on $C(\mathbb{R}_+, \Sigma)$, denoted by \mathbb{P}_x , which starts at x and is a solution of the \mathfrak{L} -martingale problem.*

The uniqueness stated in this theorem will be proved in Section 6. The existence is established in Section 7. Furthermore, we shall prove in Subsection 7.3 that $\{\mathbb{P}_x : x \in \Sigma\}$ is actually Feller continuous and defines a strong Markov process.

2.4. An absorbed diffusion. Before proceeding, we give a more precise description of the typical path under $\{\mathbb{P}_x : x \in \Sigma\}$. We start introducing the absorbing property. For each $x \in \Sigma$, denote

$$\mathcal{A}(x) := \{j \in S : x_j = 0\} \quad \text{and} \quad \mathcal{B}(x) := \mathcal{A}(x)^c.$$

For all nonempty subset $B \subseteq S$ define h_B as the first time one of the coordinates in B vanishes

$$h_B := \inf\{t \geq 0 : \prod_{j \in B} X_t(j) = 0\}. \quad (2.9)$$

Let $(\theta_t)_{t \geq 0}$ stands for the semigroup of time translation in $C(\mathbb{R}_+, \Sigma)$. Now define recursively the sequence $(\sigma_n, \mathcal{B}_n)_{n \geq 0}$ as follows. Set $\sigma_0 := 0$, $\mathcal{B}_0 := \mathcal{B}(X_0)$. For $n \geq 1$, we define

$$\sigma_n := \sigma_{n-1} + h_{\mathcal{B}_{n-1}} \circ \theta_{\sigma_{n-1}}, \quad \mathcal{B}_n := \{j \in S : X_{\sigma_n}(j) > 0\} \quad (2.10)$$

on $\{\sigma_{n-1} < \infty\}$ and $\sigma_n := \infty$, $\mathcal{B}_n := \mathcal{B}_{n-1}$ on $\{\sigma_{n-1} = \infty\}$. We also denote

$$\mathcal{A}_n := \mathcal{B}_n^c, \quad n \geq 0.$$

We shall say that a probability \mathbb{P} on $C(\mathbb{R}_+, \Sigma)$ is *absorbing* if

$$\mathbb{P}\{\mathcal{A}_n \subseteq \mathcal{A}(X_t) \text{ for all } t \geq \sigma_n\} = 1, \quad \text{for every } n \geq 0.$$

Clearly, if \mathbb{P} is absorbing then, \mathbb{P} -a.s., the sequence of subsets $(\mathcal{A}_n)_{n \geq 0}$ is increasing and

$$\exists 1 \leq n_0 \leq |\mathcal{B}_0| \quad \text{such that} \quad \sigma_{n_0} = \infty \quad \text{and} \quad \mathcal{A}_{n-1} \subsetneq \mathcal{A}_n, \quad \text{for } 1 \leq n < n_0.$$

In particular, observe that if \mathbb{P} is absorbing and starts at e_j for some $j \in S$, then

$$\mathbb{P}[X_t = e_j, \forall t \geq 0] = 1. \quad (2.11)$$

In Section 6 we prove the following result

Theorem 2.3. *For each $x \in \Sigma$, the probability \mathbb{P}_x is absorbing.*

Furthermore, we shall prove in Proposition 7.12 that

$$\text{if } x \notin \{e_j : j \in S\} \quad \text{then} \quad \mathbb{E}_x[\sigma_1] < \infty. \quad (2.12)$$

2.5. Behavior after absorption. In order to describe more precisely the evolution of the diffusion process after being absorbed at a boundary, for each $B \subseteq S$ with at least two elements, consider the simplex

$$\Sigma_B := \{x \in \mathbb{R}_+^B : \sum_{j \in B} x_j = 1\},$$

and the space $C^2(\Sigma_B)$ of functions $f : \Sigma_B \rightarrow \mathbb{R}$ which are twice-continuously differentiable. Denote by $r^B = \{r^B(j, k) : j, k \in B\}$ the jump rates of the trace of the Markov chain x_t on B . The definition of the trace of a Markov chain is recalled in Section 3.

Let $\{v_j^B : j \in B\}$ be the vectors in \mathbb{R}^B defined by

$$v_j^B := \sum_{k \in B} r^B(j, k) \{e_k - e_j\}, \quad (2.13)$$

where $\{e_j : j \in B\}$ stands for the canonical basis of \mathbb{R}^B , and let $\mathbf{b}^B : \Sigma_B \rightarrow \mathbb{R}^B$ be the vector field defined by

$$\mathbf{b}^B(x) := b \sum_{j \in B} \frac{m_j}{x_j} \mathbf{v}_j^B \mathbf{1}\{x_j > 0\}, \quad x \in \Sigma_B.$$

Denote by $\mathcal{D}_{B,A}$, $\emptyset \subsetneq A \subseteq B$, the space of functions H in $C^2(\Sigma_B)$ for which the map $x \mapsto [\mathbf{v}_j^B \cdot \nabla H(x)]/x_j$ is continuous on Σ_B for $j \in A$, and let \mathfrak{L}_B be the operator which acts on functions in $C^2(\Sigma_B)$ as in equation (2.6), but with the parameter r replaced by r^B ,

$$(\mathfrak{L}_B f)(x) := \mathbf{b}^B(x) \cdot \nabla f(x) + \frac{1}{2} \sum_{j,k \in B} m_j r^B(j,k) (\partial_{x_k} - \partial_{x_j})^2 f(x), \quad (2.14)$$

for $x \in \Sigma_B$,

Fix x in Σ and assume that $\mathcal{A}(x) = \{j \in S : x_j = 0\} \neq \emptyset$. By Theorem 2.3, \mathbb{P}_x is concentrated on trajectories X_t which belong to $C(\mathbb{R}_+, \Sigma_B)$, where $B = \mathcal{A}(x)^c$. Denote by \mathbb{P}_x^B the measure \mathbb{P}_x restricted to $C(\mathbb{R}_+, \Sigma_B)$:

$$\mathbb{P}_x^B[\mathbf{A}] = \mathbb{P}_x[\mathbf{A}], \quad \mathbf{A} \subset C(\mathbb{R}_+, \Sigma_B),$$

which is a probability measure on $C(\mathbb{R}_+, \Sigma_B)$. For each $x \in \Sigma$ and nonempty $S_0 \subseteq S$, denote by x_{S_0} the coordinates of x in S_0 .

Proposition 2.4. *Fix x in Σ and assume that $\mathcal{A}(x) = \{j \in S : x_j = 0\} \neq \emptyset$. Let $B = \mathcal{A}(x)^c$. The measure \mathbb{P}_x^B starts at x_B and solves the \mathfrak{L}_B -martingale problem (2.8) with \mathfrak{L} , \mathcal{D}_S , replaced by \mathfrak{L}_B , $\mathcal{D}_{B,B}$, respectively.*

Fix $x \in \Sigma$ and recall the definition of the absorption times σ_n introduced in (2.10). By the strong Markov property which, according to Proposition 7.11, holds for all solutions of the \mathfrak{L} -martingale problem, on the set $\{\sigma_n < \infty\}$, outside a \mathbb{P}_x -null set, any regular conditional probability distribution of \mathbb{P}_x given \mathcal{F}_{σ_n} coincides with $\mathbb{P}_{X_{\sigma_n}}$. Therefore, by Proposition 2.4, on the set $\{\sigma_n < \infty\}$, after time σ_n , X evolves on $\Sigma_{\mathcal{B}_n}$ as the diffusion with generator $\mathfrak{L}_{\mathcal{B}_n}$.

2.6. An alternative martingale problem. The previous informal description of the evolution of the process after being absorbed at the boundary can be made rigorous by the formulation of an alternative martingale problem, based on the operators $\{\mathfrak{L}_B\}$, for which $\{\mathbb{P}_x : x \in \Sigma\}$ are also solutions. To define this martingale problem, we introduce an operator \mathcal{L} and a domain $D_0(\Sigma)$. For each $B \subseteq S$ with at least two elements, let

$$\mathring{\Sigma}_B := \{x \in \Sigma_B : x_j > 0 \ \forall j \in B\}, \quad \mathring{\Sigma} := \mathring{\Sigma}_S.$$

In addition, given a function $F : \Sigma \rightarrow \mathbb{R}$ define $[F]_B : \Sigma_B \rightarrow \mathbb{R}$ as

$$[F]_B(x) := \begin{cases} F(x, \mathbf{0}), & \text{if } x \in \mathring{\Sigma}_B; \\ 0, & \text{otherwise.} \end{cases} \quad (2.15)$$

Note that B is allowed to be equal to S , in which case $[F]_S(x) = F(x) \mathbf{1}\{x \in \mathring{\Sigma}\}$. In particular, $[F]_S$ may be different from F at the boundary $\{x \in \Sigma : x_j = 0 \text{ for some } j\}$.

Define $D_0(\Sigma)$ as the set of functions $F : \Sigma \rightarrow \mathbb{R}$ such that, for all $B \subseteq S$ with at least two elements, $[F]_B$ belongs to $C^2(\Sigma_B)$ and has compact support contained in $\mathring{\Sigma}_B$. Note that functions in the domain $D_0(\Sigma)$ are not continuous.

Recall that x_{S_0} , $S_0 \subseteq S$, represents the coordinates of x in S_0 . For all $F \in D_0(\Sigma)$, define $\mathcal{L}F : \Sigma \rightarrow \mathbb{R}$ as $\mathcal{L}F(e_j) = 0$ for all $j \in S$ and

$$\mathcal{L}F(x) := (\mathfrak{L}_B[F]_B)(x_B), \quad \text{whenever } \mathcal{B}(x) = B, \quad x \notin \{e_j : j \in S\}.$$

To deal with the transitions between two consecutive time intervals in $[\sigma_{n-1}, \sigma_n)$, $n \geq 1$, consider the jump process

$$N_t := \sup\{n \geq 0 : \sigma_n \leq t\}, \quad t \geq 0,$$

and define $N_t^S := N_t \wedge |S|$, $t \geq 0$, so that $(N_t^S)_{t \geq 0}$ is a bounded right-continuous non-decreasing \mathcal{F}_t -adapted process. Clearly, if the measure \mathbb{P} is absorbing,

$$\mathbb{P}[N_t = N_t^S, \text{ for all } t \geq 0] = 1.$$

Next theorem is proved in Section 6.

Theorem 2.5. *For each $x \in \Sigma$ and any $F \in D_0(\Sigma)$,*

$$F(X_t) - \int_0^t \mathcal{L}F(X_s)ds - \int_0^t F(X_s)dN_s^S, \quad t \geq 0$$

is a \mathbb{P}_x -martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$.

The strong Markov property, assertion (2.12), Theorem 2.3 and Theorem 2.5 provide a precise picture of the dynamics determined by $\{\mathbb{P}_x : x \in \Sigma\}$.

To fix ideas assume that $x \in \tilde{\Sigma}$. By assertion (2.12), the hitting time σ_1 is \mathbb{P}_x -a.s. finite:

$$\mathbb{P}_x[\sigma_1 < \infty] = 1.$$

Before hitting the boundary of Σ , the process X_t evolves as a diffusion process with bounded and smooth coefficients \mathbf{b} and \mathbf{a}_s (see Lemma 6.2 for a more precise statement). This property characterizes the evolution on the time interval $[0, \sigma_1]$. After σ_1 , Theorem 2.3 asserts that coordinates in \mathcal{A}_1 remain equal to 0 until a new coordinate vanishes:

$$\mathbb{P}_x \left[\sum_{j \in \mathcal{A}_1} X_t(j) = 0, \sigma_1 \leq t < \sigma_2 \right] = 1.$$

It follows from Theorem 2.5 and from the strong Markov property that, given $\{\mathcal{B}_1 = B\}$, on the time interval $[\sigma_1, \sigma_2)$ the coordinates of the process X_t in B evolve as the original diffusion in which the Markov generator \mathcal{L} is replaced by \mathcal{L}_B . Furthermore, by the strong Markov property and (2.12), on $\{|\mathcal{B}| > 1\}$, the hitting time σ_2 is \mathbb{P}_x -a.s. finite. Iterating this argument, we obtain a complete description of the path under the law \mathbb{P}_x : for each $n \geq 1$, on $\{|\mathcal{B}_n| > 1\}$, σ_{n+1} is \mathbb{P}_x -a.s. finite, and on each time interval $[\sigma_n, \sigma_{n+1})$ the process X_t evolves as a diffusion on lower and lower dimensional spaces characterized by the generator $\mathcal{L}_{\mathcal{B}_n}$ where $(\mathcal{B}_n)_{n \geq 0}$ turns to be a random decreasing sequence of subsets of S . Eventually the process X_t attains a point in $\{e_j : j \in S\}$. From this time on, according to observation (2.11), the process remains trapped at this point for ever.

2.7. Remarks.

A. The case $|S| = 2$: When the set S is a pair, the diffusion X_t can be mapped to a one-dimensional diffusion. In this case, \mathcal{D} corresponds to the set of twice continuously differentiable functions $f : [0, 1] \rightarrow \mathbb{R}$, such that $f'(x) = f''(x) = 0$ for $x = 0, 1$ and the respective generator $\mathcal{L} : \mathcal{D} \rightarrow C([0, 1])$ is given by

$$(\mathcal{L}f)(x) = b \mathbf{1}\{0 < x < 1\} \left\{ \frac{M_2}{1-x} - \frac{M_1}{x} \right\} f'(x) + 2(M_1 + M_2)f''(x).$$

B. Wentzell boundary conditions: The process X_t can be viewed as a diffusion with Wentzell boundary conditions [10, Section IV.7]. In order to do that we need to introduce a differential operator for each boundary $\partial_A \Sigma := \{x \in \Sigma : \sum_{j \in A} x_j = 0, \prod_{k \in A^c} x_k > 0\}$ of Σ . It follows from the description of the process presented above that the local time of the boundary $\partial_A \Sigma$ is equal to 0 until the process hits the boundary $\partial_A \Sigma$. From this time

until one coordinate in A^c reaches 0, the local time strictly increases with slope 1, and after this latter time the local time remains constant for ever. Unfortunately, the results and the techniques on diffusions with Wentzell boundary conditions do not apply in our context because the drift explodes as the process approaches the boundary.

C. Empty sites: Our proof does not preclude the possibility that at time σ_1 more than one coordinate vanishes. We believe that this event has \mathbb{P}_x -probability equal to 0, but we were not able to exclude it, and it does not play a role in the argument.

D. Terminology: We refer to $\{\mathbb{P}_x : x \in \Sigma\}$ as an “absorbed” diffusion to distinguish it from “sticky” diffusions [10, Section IV.7]. While sticky diffusions may reflect at the boundary, even if the local time at the boundary is not identically equal to 0, as observed above the process X_t remains at the boundary once it hits it.

E. Boundary conditions: The empty coordinates remain empty due to the strong drift. The diffusivity at the boundary of Σ does not vanish. In particular, the process attempts to leave the boundary, but these attempts fail due to the strong drift which keeps the diffusion at the boundary. Actually, simulations show that there is a mesoscopic scale, between the microscopic scale of the zero-range process and the macroscopic scale of the absorbed diffusion, in which the process detaches itself from the boundary.

F. A model for concentration of wealth: The condensing zero-range processes introduced above have been used as a model to describe jamming in traffic, coalescence in granular systems, gelation in networks, and wealth concentration in macroeconomies ([6] and references therein).

G. The parameter b : It must be emphasized that the parameter b plays an important role. Condensation (cf. [9, 1, 5] for the terminology) does not occur for $b < 1$. At $b = 1$ condensation is expected to occur, but the time scale in which the condensate evolves should have logarithmic corrections. This means that for $b < 1$ the diffusion whose generator is given by (2.6), if it exists, is not expected to be absorbed at the boundary.

H. Asymptotic behavior as $L \rightarrow \infty$: As mentioned in the introduction, an interesting open problem consists in describing the evolution of condensing zero-range processes as N and $L \rightarrow \infty$ when starting from a supercritical density profile. For example, to prove the hydrodynamical behavior of the system if the initial density profile $\rho_0 : [0, 1) \rightarrow \mathbb{R}_+$ is such that $\rho_0(x) > \rho_c$ for all x , where ρ_c is the critical density (precisely defined in [9, 1, 13]). An alternative open problem, which might be more tractable, consists in proving the scaling limit of the diffusion whose generator is given by (2.6), in the case where $S = \mathbb{T}_L$ is the discrete one-dimensional torus with L points, and $r(j, k)$ the jump rates of a symmetric, nearest-neighbor random walk on \mathbb{T}_L .

2.8. The nucleation phase of condensing zero-range processes. Denote by $D(\mathbb{R}_+, \Sigma)$ the space of Σ -valued, right continuous trajectories with left limits, endowed with the Skorohod topology, and by \mathbb{P}_x^N , $x \in \Sigma_N$, the probability measure on $D(\mathbb{R}_+, \Sigma)$ induced by the Markov chain X_t^N starting from x . Expectation with respect to \mathbb{P}_x^N is represented by \mathbb{E}_x^N .

Theorem 2.6. *Let $x_N \in \Sigma_N$ be a sequence converging to $x \in \Sigma$. Then, $\mathbb{P}_{x_N}^N$ converges to \mathbb{P}_x in the Skorohod topology.*

Note that for each $x \in \Sigma$, the measure \mathbb{P}_x is concentrated on the space $C(\mathbb{R}_+, \Sigma)$ of continuous trajectories.

The proof of Theorem 2.6 is divided in two steps. We show in Proposition 7.6 that the sequence of probability measures $\mathbb{P}_{x_N}^N$ is tight, and we prove in Proposition 7.7 that

any limit point of the sequence $\mathbb{P}_{x_N}^N$ solves the martingale problem (2.8). Of course, the existence part of Theorem 2.2 follows from this result. It is worth remarking that in the proof of tightness we do not need to require $b > 1$.

3. HARMONIC EXTENSION

Fix a proper subset B of S with at least two elements and let $A = B^c$. The main result of this section asserts that it is possible to extend a smooth function $f : \Sigma_B \rightarrow \mathbb{R}$ to a function $F : \Sigma \rightarrow \mathbb{R}$ in such a way that F belongs to \mathcal{D}_A and $(\mathcal{L}F)(x) = (\mathcal{L}_B f)(x_B)$ for x in the submanifold

$$\Sigma_{B,0} := \{x \in \Sigma : \sum_{i \in A} x_i = 0\}.$$

3.A The Trace process. Let us start recalling the definition of the trace of a Markov chain on a subset of its state space. We refer to [4] for more details.

Let $D(\mathbb{R}_+, S)$ be the set of right-continuous trajectories $e : \mathbb{R}_+ \rightarrow S$ with left limits, endowed with the Skorohod topology. Denote by \mathbf{P}_j , $j \in S$, the probability measure on $D(\mathbb{R}_+, S)$ induced by the Markov chain $(x_t)_{t \geq 0}$ with jump rates $\mathbf{r} = \{r(j, k) : j, k \in S\}$ and starting from j . Denote by T_{S_0} (resp. $T_{S_0}^+$), $S_0 \subseteq S$, the hitting time of (resp. return time to) S_0 :

$$T_{S_0} = \inf\{t \geq 0 : x_t \in S_0\}, \quad T_{S_0}^+ = \inf\{t \geq \tau_1 : x_t \in S_0\},$$

where τ_1 represents the time of the first jump: $\tau_1 = \inf\{t \geq 0 : x_t \neq x_0\}$. Fix a nonempty subset B of S with at least two elements and denote by $(x_t^B)_{t \geq 0}$ the trace of $(x_t)_{t \geq 0}$ on B . This is the irreducible, B -valued Markov chain whose jump rates, denoted by $\mathbf{r}^B = \{r^B(j, k) : j, k \in B\}$, are given by

$$r^B(j, k) := \lambda(j) \mathbf{P}_j[T_k = T_B^+], \quad k \neq j \in B, \quad (3.1)$$

and set $r^B(j, j) = 0$, $j \in S$ for notational convenience.

3.B The functions \mathbf{u}_k . For each $k \in B$, let $\mathbf{u}_k = \mathbf{u}_k^B : S \rightarrow [0, 1]$ be the only \mathcal{L} -harmonic extension on S of the indicator of $\{k\}$ on B , i.e. \mathbf{u}_k is the solution of

$$\begin{cases} \mathbf{u}_k(j) = \delta_{j,k}, & \text{for } j \in B; \\ \mathcal{L}\mathbf{u}_k(j) = 0, & \text{for } j \in S \setminus B. \end{cases}$$

Actually, the vectors $\{\mathbf{u}_k : k \in B\}$ can also be written as probabilities:

$$\mathbf{P}_j[T_k = T_B] = \mathbf{u}_k(j), \quad \forall k \in B, j \in S. \quad (3.2)$$

In particular, by using the strong Markov property in (3.1) we get the relation

$$r^B(j, k) = r(j, k) + \sum_{\ell \in B^c} r(j, \ell) \mathbf{P}_\ell[T_k = T_B] = \sum_{\ell \in S} r(j, \ell) \mathbf{u}_k(\ell), \quad \text{for } k \neq j \in B.$$

3.C Relation between \mathbf{u}_k and \mathbf{v}_j^B . Recall the definition of the vectors $\{\mathbf{v}_j^B : j \in B\}$ introduced in (2.13). We claim that

$$\mathbf{v}_j^B(k) = \mathcal{L}\mathbf{u}_k(j) \quad \text{for all } j, k \in B. \quad (3.3)$$

Indeed, on the one hand, by definition of \mathbf{v}_j^B , and by the last identity of the previous subsection,

$$\mathbf{v}_j^B(k) = r^B(j, k) = \mathcal{L}\mathbf{u}_k(j), \quad \text{for } k \neq j \in B.$$

On the other hand, for any $k \in B$,

$$\sum_{j \in B} m_j \mathbf{v}_j^B(k) = 0 = \sum_{j \in B} m_j \mathcal{L}\mathbf{u}_k(j).$$

The first identity follows from the fact that \mathbf{m} restricted to B is also an invariant measure for \mathbf{r}^B . For the second equality, as $\mathcal{L}\mathbf{u}_k(j) = 0$ for $j \notin B$, observe that $\sum_{j \in B} m_j \mathcal{L}\mathbf{u}_k(j) = \sum_{j \in S} m_j \mathcal{L}\mathbf{u}_k(j) = 0$ because \mathbf{m} is an invariant measure for \mathbf{r} . The two previous displayed equations yield claim (3.3).

Let \mathcal{L}^B stand for the generator corresponding to the jump rates \mathbf{r}^B . Then, for any $j, k \in B$, $\mathbf{v}_j^B(k)$ equals $\mathcal{L}^B \mathbf{e}_k(j)$, where to keep notation simple, we let $\{\mathbf{e}_k : k \in B\}$ stand for the canonical basis of \mathbb{R}^B . Thus, (3.3) can also be written as

$$\mathcal{L}^B \mathbf{e}_k \equiv \mathcal{L}\mathbf{u}_k \quad \text{on } B \text{ for any } k \in B. \quad (3.4)$$

3.D The projection Υ^B . Recall the definition of the submanifold $\Sigma_{B,0}$ introduced at the beginning of this section. Denote by $\Upsilon = \Upsilon^B : \Sigma \rightarrow \Sigma_B$ the linear map given by

$$[\Upsilon(x)]_k = \mathbf{u}_k \cdot x = x_k + \sum_{j \in A} \mathbf{u}_k(j) x_j, \quad k \in B. \quad (3.5)$$

It is easy to check that $\Upsilon(x) \in \Sigma_B$ for all $x \in \Sigma$, and that

$$\Upsilon(x) = x_B \quad \text{for all } x \in \Sigma_{B,0},$$

where x_B stands for the coordinates of x in B .

We claim that

$$\Upsilon(\mathbf{v}_j) = \mathbf{v}_j^B, \quad j \in B. \quad (3.6)$$

Indeed, by definition of Υ and of the vectors $\{\mathbf{v}_j : j \in S\}$,

$$\Upsilon(\mathbf{v}_j)(k) = \mathbf{u}_k \cdot \mathbf{v}_j = \mathcal{L}\mathbf{u}_k(j), \quad j \in S, k \in B. \quad (3.7)$$

By (3.3), the last expression equals $\mathbf{v}_j^B(k)$ proving the assertion.

3.E Harmonic extensions. For a nonempty subset B of S , denote by \mathbf{m}_B the restriction of the measure \mathbf{m} on B : $m_B(j) = m(j)$, $j \in B$. Denote by \mathcal{L}^* (resp. $\mathcal{L}^{B,*}$) the adjoint of the generator \mathcal{L} (resp. \mathcal{L}^B) in $L^2(\mathbf{m})$ (resp. $L^2(\mathbf{m}_B)$), and by \mathcal{S} (resp. \mathcal{S}^B) the symmetric part of \mathcal{L} (resp. \mathcal{L}^B): $\mathcal{S} = (1/2)(\mathcal{L} + \mathcal{L}^*)$ (resp. $\mathcal{S}^B = (1/2)(\mathcal{L}^B + \mathcal{L}^{B,*})$).

Given a function $f \in C^2(\Sigma_B)$, define $F \in C^2(\Sigma)$ as $F = f \circ \Upsilon$.

Lemma 3.1. *The function F is an extension of f in the sense that*

$$F(x) = f(x_B), \quad \forall x \in \Sigma_{B,0}. \quad (3.8)$$

Moreover,

$$\mathbf{v}_j \cdot \nabla F(x) = 0, \quad j \in A, x \in \Sigma, \quad (3.9)$$

so that $F \in \mathcal{D}_A$. Finally,

$$\mathfrak{L}F(x) = \mathfrak{L}_B f(x_B), \quad x \in \mathring{\Sigma}_{B,0}, \quad (3.10)$$

where

$$\mathring{\Sigma}_{B,0} := \{x \in \Sigma_{B,0} : x_j > 0 \text{ } j \in B\}.$$

Proof. The first assertion of the lemma follows from the displayed equation below (3.5). We turn to the second assertion. Fix an arbitrary $x \in \Sigma$. By definition of Υ , $\partial_{x_k} \Upsilon(x)(\ell) = \mathbf{u}_\ell(k)$ for all $\ell \in B$, $k \in S$. Hence, by definition of F and by (3.7),

$$\mathbf{v}_j \cdot \nabla F(x) = \Upsilon(\mathbf{v}_j) \cdot [(\nabla f)(\Upsilon(x))], \quad j \in S. \quad (3.11)$$

If j belongs to A , by (3.7) and by definition of \mathbf{u}_k , $\Upsilon(\mathbf{v}_j)(k) = \mathcal{L}\mathbf{u}_k(j) = 0$ for all $k \in B$. This completes the proof of the second assertion.

We turn to the proof of the last assertion of the lemma. Fix $x \in \overset{\circ}{\Sigma}_{B,0}$ and so $\Upsilon(x) = x_B$. We first examine the first-order terms. By (3.11) and (3.6) we have

$$\mathbf{v}_j \cdot \nabla F(x) = \mathbf{v}_j^B \cdot \nabla f(x_B), \quad j \in B.$$

Therefore, by (3.9) and this last identity we conclude that the first-order part of $(\mathfrak{L}F)(x)$ equals the first-order part of $(\mathfrak{L}_B f)(x_B)$.

It remains to examine the second-order terms of the generators. The second-order piece of $(\mathfrak{L}F)(x)$ is

$$\frac{1}{2} \sum_{j,k \in S} m_j r(j,k) (\partial_{x_k} - \partial_{x_j})^2 (f \circ \Upsilon)(x).$$

A simple computation shows that

$$(\partial_{x_k} - \partial_{x_j})^2 (f \circ \Upsilon)(x) = \sum_{i,\ell \in B} \partial_{x_\ell, x_i}^2 f(x_B) \{\mathbf{u}_\ell(k) - \mathbf{u}_\ell(j)\} \{\mathbf{u}_i(k) - \mathbf{u}_i(j)\}.$$

In view of this identity, interchanging the sums, the penultimate displayed equation becomes

$$\sum_{i,\ell \in B} (\partial_{x_\ell, x_i}^2 f)(x_B) \langle \mathbf{u}_\ell, -\mathcal{S} \mathbf{u}_i \rangle_{\mathbf{m}}, \quad (3.12)$$

where, recall, \mathcal{S} represents the symmetric part of the generator \mathcal{L} in $L^2(\mathbf{m})$.

Fix $i, \ell \in B$. Since $\mathcal{L} \mathbf{u}_i$ vanishes on $A = B^c$,

$$\langle \mathbf{u}_\ell, -\mathcal{L} \mathbf{u}_i \rangle_{\mathbf{m}} = \sum_{j \in B} \mathbf{u}_\ell(j) (-\mathcal{L} \mathbf{u}_i)(j) \mathbf{m}(j)$$

On the set B , \mathbf{u}_ℓ and \mathbf{e}_ℓ coincide, while, by (3.4), $\mathcal{L} \mathbf{u}_i = \mathcal{L}^B \mathbf{e}_i$. Hence, the penultimate formula is equal to

$$\sum_{i,\ell \in B} (\partial_{x_\ell, x_i}^2 f)(x_B) \langle \mathbf{e}_\ell, -\mathcal{S}^B \mathbf{e}_i \rangle_{\mathbf{m}_B},$$

where the scalar product is now performed over B . This expression is equal to

$$\frac{1}{2} \sum_{i,\ell \in B} \partial_{x_\ell, x_i}^2 f(x_B) \sum_{j,k \in B} m_j r^B(j,k) \{\mathbf{e}_\ell(k) - \mathbf{e}_\ell(j)\} \{\mathbf{e}_i(k) - \mathbf{e}_i(j)\}.$$

By interchanging the sums, this term becomes

$$\frac{1}{2} \sum_{j,k \in B} m_j r^B(j,k) (\partial_{x_k} - \partial_{x_j})^2 f(x_B).$$

This is exactly the second-order term of $(\mathfrak{L}_B f)(x_B)$. Last assertion of Lemma 3.1 is hence proved. \square

4. THE DOMAIN OF THE GENERATOR

The proof that all solutions of the \mathfrak{L} -martingale problem are absorbed at the boundary, relies on the existence of super-harmonic, non-negative functions which are strictly positive at the boundary. The goal of this section is to provide such functions.

This is achieved by introducing in (4.6) a class of non-negative functions and by applying Lemmata 4.3 and 4.4. Lemma 4.3 states that it is possible to extend certain functions $f : \Sigma_B \rightarrow \mathbb{R}$ which belong to \mathcal{D}_D , $D \subset B$, to functions $F : \Sigma \rightarrow \mathbb{R}$ which belong to $\mathcal{D}_{D \cup B^c}$, while Lemma 4.4 states that it is possible to modify a function $F : \Sigma \rightarrow \mathbb{R}$ which belongs to \mathcal{D}_A in a neighborhood of the set $\{x \in \Sigma : \prod_{j \in A^c} x_j = 0\}$ to transform it into a function which belongs to \mathcal{D}_S .

We start in Lemma 4.1 below by defining a class of functions I_A , $\emptyset \subsetneq A \subsetneq S$, which belong to \mathcal{D}_A . These functions play a key role in the argument and they have to be interpreted as smooth perturbations of the maps $x \mapsto \sum_{i \in A} x_i^2$.

For a nonempty subset S_0 of S , let

$$\|x\|_{S_0} := \left(\sum_{j \in S_0} x_j^2 \right)^{1/2}, \quad x \in \Sigma,$$

and set $\|\cdot\| := \|\cdot\|_S$.

Lemma 4.1. *Let A be a nonempty, proper subset of S . There exists a nonnegative, smooth function $I_A : \Sigma \rightarrow \mathbb{R}$ in \mathcal{D}_A , and constants $0 < c_1 < C_1 < \infty$, such that for all $x \in \Sigma$,*

$$c_1 \|x\|_A \leq \sqrt{I_A(x)} \leq C_1 \|x\|_A. \quad (4.1)$$

Furthermore, for $x, y \in \Sigma$,

$$I_A(y) = \alpha^2 I_A(x) \text{ if } y_A = \alpha x_A \text{ for some } \alpha \geq 0. \quad (4.2)$$

In particular, $I_A(x)$ only depends on x_A for all $x \in \Sigma$.

The proof of this lemma is postponed to the last section of this article. The function $I_A(x)$ has to be understood as a perturbation of the function $x \mapsto \|x\|_A^2$ to turn this latter function an element of \mathcal{D}_A . Let

$$J_A(x) = \sqrt{I_A(x)}, \quad x \in \Sigma, \quad (4.3)$$

Of course, J_A is smooth on $\{x \in \Sigma : \|x\|_A > 0\}$. Furthermore, by (4.2), for $x, y \in \Sigma$ such that $x_A \neq 0$, $y_A = \alpha x_A$ for some $\alpha > 0$,

$$\nabla J_A(y) = \nabla J_A(x) \quad \text{and} \quad \|y\|_A \text{Hess } J_A(y) = \|x\|_A \text{Hess } J_A(x). \quad (4.4)$$

In particular,

$$\sup_{\|x\|_A > 0} \|\nabla J_A(x)\| < \infty \quad \text{and} \quad \sup_{\|x\|_A > 0} \|x\|_A |\partial_{x_j, x_k}^2 J_A(x)| < \infty, \quad j, k \in S. \quad (4.5)$$

We shall use the following estimate in Lemma 4.3 below.

Lemma 4.2. *For all $k \in A$,*

$$\sup_{\|x\|_A > 0} \frac{|\mathbf{v}_k \cdot \nabla J_A(x)|}{x_k} \|x\|_A < \infty.$$

Proof. Fix $k \in A$. For every $x \in \Sigma$ such that $\|x\|_A > 0$,

$$\sum_{j \in A} x(j) < 2|A|^{1/2} \|x\|_A.$$

In particular, for each $x \in \Sigma$ such that $\|x\|_A > 0$ there exists some $z \in \Sigma$ such that

$$z_A = \frac{x_A}{2|A|^{1/2} \|x\|_A}.$$

In view of (4.4), for this choice we have that

$$\frac{|\mathbf{v}_k \cdot \nabla J_A(x)|}{x_k} \|x\|_A = \frac{1}{2|A|^{1/2}} \frac{|\mathbf{v}_k \cdot \nabla J_A(z)|}{z_k}.$$

Therefore,

$$\begin{aligned}
\sup_{\|x\|_A > 0} \frac{|\mathbf{v}_k \cdot \nabla J_A(x)|}{x_k} \|x\|_A &\leq \frac{1}{2|A|^{1/2}} \sup_{\|z\|_A = (4|A|)^{-1/2}} \frac{|\mathbf{v}_k \cdot \nabla J_A(z)|}{z_k} \\
&= \frac{1}{2|A|^{1/2}} \sup_{\|z\|_A = (4|A|)^{-1/2}} \frac{|\mathbf{v}_k \cdot \nabla I_A(z)|}{2J_A(z) z_k} \\
&\leq \frac{1}{2c_1} \sup_{z_k \neq 0} \frac{|\mathbf{v}_k \cdot \nabla I_A(z)|}{z_k},
\end{aligned}$$

where we used estimate (4.1) in the last inequality. The last expression is finite because $I_A \in \mathcal{D}_A$, which completes the proof. \square

Fix a nonempty subset B of S and let $A = B^c$. Suppose now that in Lemma 3.1, $f : \Sigma_B \rightarrow \mathbb{R}$ is of the special form

$$f(x) = \prod_{j \in D} x_j^{p+1}, \quad x \in \Sigma_B, \quad (4.6)$$

for some nonempty subset D of B and for some $p > 1$. In this case, we may improve Lemma 3.1 obtaining an extension F of f which belongs to $\mathcal{D}_{A \cup D}$. From now on, for each nonempty subset S_0 of S , let

$$\pi_{S_0}(x) = \prod_{j \in S_0} x_j. \quad (4.7)$$

Lemma 4.3. *Let $f : \Sigma_B \rightarrow \mathbb{R}$ be given by (4.6) for a nonempty subset D of B and $p > 1$. Then there exists a function $F : \Sigma \rightarrow \mathbb{R}$ in $\mathcal{D}_{A \cup D}$ satisfying (3.8) and (3.10). Furthermore, if $\pi_D(x) = 0$ then $\mathfrak{L}F(x) = 0$.*

Proof. If $A = \emptyset$, it is easy to check that $F = f$ satisfies all the requirements. We then assume A is nonempty. Let $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ be a non-increasing function in $C^2(\mathbb{R})$ which is equal to 1 on $(-\infty, 0]$, and is equal to 0 on $[1, \infty)$. For example, the function which on the interval $[0, 1]$ is given by

$$\Psi(a) = (1-a)^3(1+3a+6a^2), \quad a \in [0, 1]. \quad (4.8)$$

We fix the constant

$$\beta := \frac{2(1+|A|^{1/2})}{c_1},$$

where $c_1 > 0$ is the constant given in (4.1).

Let $F : \Sigma \rightarrow \mathbb{R}$ be defined by

$$F(x) = \begin{cases} (f \circ \Upsilon)(x) \Psi\left(\frac{\beta J_A(x)}{\pi_D(\Upsilon(x))} - 1\right), & \text{if } \pi_D(\Upsilon(x)) > 0; \\ 0, & \text{otherwise,} \end{cases}$$

where $\Upsilon : \Sigma \rightarrow \Sigma_B$ is the linear map introduced in (3.5). Note that

$$F(x) = (f \circ \Upsilon)(x) \quad \text{if } \pi_D(\Upsilon(x)) = 0$$

because both expressions vanish when $\pi_D(\Upsilon(x)) = 0$.

From the definition of Ψ it easily follows that

$$F(x) = f(\Upsilon(x)) = f(x_B), \quad \text{for all } x \in \Sigma_{B,0},$$

proving that F satisfies (3.8).

A. F belongs to $C^1(\Sigma)$. Denote by \mathcal{V} the open subset $\{x \in \Sigma : \pi_D(\Upsilon(x)) > 0\}$ and let

$$R(x) = \beta J_A(x)/\pi_D(\Upsilon(x)) \quad \text{for } x \in \mathcal{V}.$$

It is simple to check that $\Psi(R-1)$ and F are of class C^2 in \mathcal{V} . In particular, to prove that F belongs to $C^2(\Sigma)$, we only need to examine the behavior of the derivatives of F close to the boundary of \mathcal{V} .

We claim that

$$\|\nabla F(x)\| \leq C \left\{ \|\nabla f(w)\|_B + \pi_D(w)^p \right\}, \quad x \in \mathcal{V}, \quad (4.9)$$

where w stands for $\Upsilon(x)$, and C for a finite positive constant, independent of x , and which may be different from line to line.

On the one hand, since $\Psi \equiv 1$ on $(-\infty, 0]$, by (4.1), it is clear that

$$\Psi(R-1) \equiv 1 \text{ on the open subset } \{x \in \Sigma : \beta C_1 \|x\|_A < \pi_D(\Upsilon(x))\},$$

so that (4.9) holds in this open subset of \mathcal{V} . On the other hand, if x is a point in \mathcal{V} such that $\|x\|_A > 0$, by (4.5) and by the fact that $\Psi'(R(x)-1) = 0$ for $R(x) \geq 2$,

$$\|\nabla\{\Psi(R-1)\}(x)\| \leq \frac{C}{\pi_D(\Upsilon(x))}, \quad \forall x \in \mathcal{V},$$

which proves that (4.9) also holds in this case, in view of the definition of f .

By (4.9) and by the definition of f , $\nabla F(x)$ vanishes as $x \in \mathcal{V}$ approaches the boundary of \mathcal{V} . On the other hand, since Ψ is bounded, it follows from the definition of f that

$$\partial_{x_j} F(x) = 0, \quad \forall j \in S, \quad x \in \Sigma \setminus \mathcal{V}, \quad (4.10)$$

which proves that F belongs to $C^1(\Sigma)$.

B. F belongs to $C^2(\Sigma)$. We claim that there exists a finite constant C such that for all $j, k \in S$, and all $x \in \mathcal{V}$,

$$|(\partial_{x_j x_k}^2 F)(x)| \leq C \pi_D(\Upsilon(x))^{p-1}. \quad (4.11)$$

Indeed, for x in the open set $\{x \in \Sigma : \beta C_1 \|x\|_A < \pi_D(\Upsilon(x))\}$, $F(x) = f(\Upsilon(x))$ and the assertion is easily proved. Additionally, for $x \in \mathcal{V}$ such that $\|x\|_A > 0$, by (4.1), by (4.5), and by the fact that $\Psi'(R(x)-1) = \Psi''(R(x)-1) = 0$ for $R(x) \geq 2$ and for $R(x) \leq 1$,

$$\partial_{x_j x_k}^2 \{\Psi(R-1)\}(x) \leq \frac{C}{\pi_D(\Upsilon(x))^2}.$$

Assertion (4.11) for $x \in \mathcal{V}$ such that $\|x\|_A > 0$ is a simple consequence of this estimate, of the bound on the first derivative of $\Psi(R-1)$ obtained in part A of the proof, and of the definition of f .

We claim that

$$\partial_{x_j x_k}^2 F(x) = 0, \quad \forall j, k \in S, \quad x \in \Sigma \setminus \mathcal{V}. \quad (4.12)$$

Indeed, fix $x_0 \in \Sigma$ such that $\pi_D(\Upsilon(x_0)) = 0$, so that $F(x_0) = \partial_{x_j} F(x_0) = 0$ for all $j \in S$. By (4.9) and (4.10) we have

$$\frac{\|\nabla F(x)\|}{\|x - x_0\|} \leq C \left\{ \frac{\|(\nabla f)(\Upsilon(x))\|_B}{\|x - x_0\|} + \frac{\pi_D(\Upsilon(x))^p}{\|x - x_0\|} \right\}, \quad \text{for } x \neq x_0.$$

Since $\nabla f(w)$, $\pi_D(w)^p$, $\text{Hess } f(w)$, and $\nabla \pi_D(w)^p$ vanish at $w = \Upsilon(x_0)$,

$$\|(\nabla f)(\Upsilon(x))\|_B + \pi_D(\Upsilon(x))^p \leq C \|\Upsilon(x) - \Upsilon(x_0)\|_B^2 \leq C \|x - x_0\|^2,$$

for all $x \in \Sigma$, which proves (4.12), and, in view of (4.11), that F belongs to $C^2(\Sigma)$.

C. F satisfies (3.10). In order to prove this property, observe that the functions F and $f \circ \Upsilon$ coincide on

$$\{x \in \Sigma : \beta J_A(x) < \pi_D(\Upsilon(x))\}.$$

Since J_A and $\pi_D \circ \Upsilon$ are continuous functions, this is an open subset of Σ . Moreover, $\overset{\circ}{\Sigma}_{B,0}$ is contained in this open subset. Therefore, for any $x \in \overset{\circ}{\Sigma}_{B,0}$, $\mathfrak{L}F(x) = \mathfrak{L}(f \circ \Upsilon)(x)$. By Lemma 3.1, this latter term is equal to $\mathfrak{L}_B f(x_B)$.

D. F belongs to \mathcal{D}_D . By (4.10), $\mathbf{v}_\ell \cdot \nabla F(x) = 0$ for $x \in \Sigma \setminus \mathcal{V}$ and $\ell \in S$. Thus, in the proof that F belongs to $\mathcal{D}_{D \cup A}$, in the limit appearing in (2.5), we only need to consider points x in \mathcal{V} . This is assumed below and in the Step **E** without further comment.

Fix $j \in D$. By (4.1) and by Cauchy-Schwarz inequality in \mathbb{R}^A ,

$$R(x) \geq \frac{\beta J_A(x)}{\Upsilon(x)(j)} \geq \frac{\beta c_1 \|x\|_A}{x_j + \sum_{k \in A} \mathbf{u}_j(k) x_k} \geq \frac{\beta c_1 \|x\|_A}{x_j + |A|^{1/2} \|x\|_A}. \quad (4.13)$$

Hence, by definition of β , $R(x) > \beta c_1 / [1 + |A|^{1/2}] = 2$ for x such that $x_j < \|x\|_A$. In particular, by definition of Ψ ,

$$F \equiv 0 \text{ on the open subset } \{x \in \Sigma : x_j < \|x\|_A\}.$$

It follows from this observation and from (4.9) that

$$|\mathbf{v}_j \cdot \nabla F(x)| \leq C(\|\nabla f(w)\|_B + \pi_D(w)^p) \mathbf{1}\{\|x\|_A \leq x_j\}, \quad x \in \mathcal{V},$$

where $w = \Upsilon(x)$. For $x \in \Sigma$ such that $\|x\|_A \leq x_j$,

$$w_j := \Upsilon(x)(j) \leq x_j + |A|^{1/2} \|x\|_A \leq (1 + |A|^{1/2}) x_j.$$

Therefore, by the next to the last displayed formula and by definition of f ,

$$\begin{aligned} \frac{|\mathbf{v}_j \cdot \nabla F(x)|}{x_j} &\leq C \left\{ \frac{\|\nabla f(w)\|_B}{w_j} + \frac{\pi_D(w)^p}{w_j} \right\} \mathbf{1}\{\|x\|_A \leq x_j\} \\ &\leq C \pi_D(w)^{p-1} \mathbf{1}\{\|x\|_A \leq x_j\}, \quad x \in \mathcal{V}, \quad x_j \neq 0. \end{aligned}$$

Fix $y \in \Sigma$ such that $y_j = 0$. If $\|y\|_A > 0$, in view of the indicator in the previous estimate and by the remark formulated at the beginning of this step,

$$\lim_{\substack{x \rightarrow y \\ x_j > 0}} \frac{|\mathbf{v}_j \cdot \nabla F(x)|}{x_j} = 0.$$

In contrast, if $\|y\|_A = 0$, $\pi_D(\Upsilon(y)) = 0$ because $\Upsilon(y)(j) = 0$. Hence, the same conclusion holds because $\pi_D(w)$ converges to $\pi_D(\Upsilon(y))$ and $p > 1$. This concludes the proof that F belongs to \mathcal{D}_D .

E. F belongs to \mathcal{D}_A . Recall from the previous step that we may restrict our analysis to points x in \mathcal{V} . Fix $k \in A$ and $y \in \Sigma$ such that $y_k = 0$.

Identity (3.9) for the functions $\pi_D \circ \Upsilon$ and $f \circ \Upsilon$ yield that

$$\mathbf{v}_k \cdot \nabla F(x) = \beta \pi_D(\Upsilon(x))^p \Psi'(R(x) - 1) \mathbf{v}_k \cdot \nabla J_A(x),$$

for all $x \in \mathcal{V}$ such that $\|x\|_A > 0$.

We consider separately three cases which all rely on the identity appearing in the previous displayed formula. Assume first that $\|y\|_A > 0$. In this case, since $\nabla J_A(x) = \nabla I_A(x)/2J_A(x)$, and since I_A belongs to \mathcal{D}_A , in view of (4.1),

$$\lim_{\substack{x \rightarrow y \\ x_k > 0}} \frac{\mathbf{v}_k \cdot \nabla F(x)}{x_k} = 0.$$

Next, assume that $\|y\|_A = 0$ and that $\pi_D(\Upsilon(y)) > 0$. In this case, $\Psi'(R(x) - 1) = 0$ in a neighborhood of y , so that $v_k \cdot \nabla F$ vanishes in a neighborhood of y in \mathcal{V} .

It remains to consider the case in which $\|y\|_A = 0$ and $\pi_D(\Upsilon(y)) = 0$. For $x \in \mathcal{V}$ such that $x_k > 0$, by the identity appearing in the first displayed equation of this step, by Lemma 4.2, and by definition of Ψ ,

$$\left| \frac{v_k \cdot \nabla F(x)}{x_k} \right| \leq C \frac{\pi_D(\Upsilon(x))^p}{\|x\|_A} \mathbf{1}\{R(x) \geq 1\} \leq C \pi_D(\Upsilon(x))^{p-1},$$

where we used estimate (4.1) and the definition of R in the last inequality. As $x \rightarrow y$, the right hand side converges to $\pi_D(\Upsilon(y))^{p-1} = 0$ because $p > 1$. This completes the proof that F belongs to \mathcal{D}_A .

F. $\mathfrak{L}F(x) = 0$ if $\pi_D(x) = 0$. Fix $x \in \Sigma$ such that $\pi_D(x) = 0$. If $\|x\|_A = 0$ then $\pi_D(\Upsilon(x)) = \pi_D(x) = 0$ and so $\mathfrak{L}F(x) = 0$ in view of (4.10) and (4.12). If $\|x\|_A > 0$, F vanishes in a neighborhood of x because so thus $\Psi(R(x) - 1)$. In particular, $\mathfrak{L}F(x) = 0$. \square

We conclude this section by proving in Lemma 4.4 below that a function F in \mathcal{D}_A , $\emptyset \subsetneq A \subsetneq S$, can be slightly modified into a function H in \mathcal{D}_S . For each nonempty subset $S_0 \subseteq S$ and $\epsilon > 0$, let

$$\Lambda_{S_0}(\epsilon) := \{x \in \Sigma : \min_{j \in S_0} x_j \geq \epsilon\}. \quad (4.14)$$

Lemma 4.4. *Fix a nonempty, proper subset A of S and a function F in \mathcal{D}_A . For every $\epsilon > 0$ there exists a function H in \mathcal{D}_S such that*

$$F(x) = H(x) \quad \text{and} \quad \mathfrak{L}F(x) = \mathfrak{L}H(x), \quad x \in \Lambda_B(\epsilon),$$

where $B = A^c$. Moreover, if $F \in \mathcal{D}_A$ is such that $[F]_B$ has a compact support contained in Σ_B then, there exists some $\epsilon > 0$ and $H \in \mathcal{D}_S$ satisfying the previous identities and such that

$$H(x) = F(x), \quad x \in \Sigma_{B,0}.$$

Proof. Recall function Ψ defined in (4.8). Let $\Phi(r) = 1 - \Psi(r/3)$, $r \in \mathbb{R}$, so that Φ is a non-decreasing C^2 functions such that $\Phi(r) = 0$ for $r \leq 0$, and $\Phi(r) = 1$ for $r \geq 3$. Let $\alpha = c_1/4C_1$, where c_1, C_1 have been introduced in (4.1) and fix some arbitrary $\epsilon > 0$. For $k \in B$, let $G_k : \Sigma \rightarrow \mathbb{R}$, be given by

$$G_k(x) = \phi_{k,\emptyset}(x) \prod_{\substack{D \subseteq A, \\ |D|=1}} \phi_{k,D}(x) \prod_{\substack{D \subseteq A, \\ |D|=2}} \phi_{k,D}(x) \cdots \phi_{k,A}(x).$$

In this formula,

$$\phi_{k,D}(x) = \Phi\left(\frac{9\alpha^{2|D|} I_{D \cup \{k\}}(x)}{\epsilon^2} - 1\right), \quad \text{for each } \emptyset \subseteq D \subseteq A.$$

The proof of the lemma relies on the elementary properties of the functions G_k and $\phi_{k,D}$ listed below. Since $J_k(x) \leq C_1 x_k$, $\phi_{k,\emptyset}(x) = 0$ for $x_k \leq \epsilon/3C_1$. Thus,

$$G_k(x) = 0 \quad \text{for} \quad x_k \leq \epsilon/3C_1. \quad (4.15)$$

On the other hand, by (4.1), $J_{D \cup \{k\}}(x) \geq c_1 x_k$. Hence, since $\Phi(r) = 1$ for $r \geq 3$ and since $\alpha \leq 1$,

$$G_k(x) = 1 \quad \text{for} \quad x_k \geq \epsilon/c_1 \alpha^{|A|}. \quad (4.16)$$

By similar reasons, we have

$$\nabla \phi_{k,D}(x) = 0 \quad \text{if} \quad 3\alpha^{|D|} c_1 \|x\|_{D \cup \{k\}} \geq 2\epsilon. \quad (4.17)$$

Finally, since $\Phi(r) = 0$ for $r \leq 0$, by (4.1),

$$\phi_{k,D}(x) = 0 \quad \text{if} \quad 3\alpha^{|D|}C_1\|x\|_{D \cup \{k\}} \leq \epsilon. \quad (4.18)$$

Let $G : \Sigma \rightarrow \mathbb{R}$ be defined by

$$G(x) = \prod_{k \in B} G_k(x), \quad x \in \Sigma. \quad (4.19)$$

We claim that

$$H(x) = F(x)G(x), \quad x \in \Sigma, \quad (4.20)$$

fulfills all the properties required in the lemma. It follows from (4.16) that H and F coincide on the set $\Lambda_B(\delta)$, where $\delta = \epsilon/c_1\alpha^{|A|}$. Since $\epsilon > 0$ is arbitrary, this proves the first assertion of the lemma. By (4.15), H belongs to \mathcal{D}_B . It remains to show that H belongs to \mathcal{D}_A .

Fix $j \in A$. It is clear that $F_1 F_2$ belongs to \mathcal{D}_j if both functions belong. It is also clear that $\Xi(F)$ belongs to \mathcal{D}_j if F belongs to \mathcal{D}_j and if $\Xi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function. Therefore, as F and I_C belong to \mathcal{D}_j if the set C contains j , to prove that H belongs to \mathcal{D}_j it is enough to show that all terms which do not contain x_j in the product in (4.19) belong to \mathcal{D}_j . A general term in such product has the form $\phi_{k,D}(x)$, where D is a proper subset A which does not contain x_j and $k \in B$. We will not prove that $\phi_{k,D}(x)$ belongs to \mathcal{D}_j , but that $\phi_{k,D \cup \{j\}}(x) [\mathbf{v}_j \cdot \nabla \phi_{k,D}(x)]$ vanishes for x_j small enough. Indeed, by (4.17) and (4.18), this product vanishes unless

$$3\alpha^n c_1 \|x\|_{D \cup \{k\}} \leq 2\epsilon \quad \text{and} \quad 3\alpha^{n+1} C_1 \|x\|_{D \cup \{k,j\}} \geq \epsilon,$$

where $|D| = n$. It follows from these inequalities and from the definition of α that $x_j \geq \epsilon/2$. This completes the proof of the second assertion of the lemma.

We turn to the last assertion. Assume that $[F]_B$, introduced in (2.15), has a compact support contained in $\overset{\circ}{\Sigma}_B$. There exists therefore $\epsilon_0 > 0$ such that

$$F(x) = 0, \quad x \in \Sigma_{B,0} \setminus \Lambda_B(\epsilon_0).$$

Let $\epsilon = \epsilon_0 c_1 \alpha^{|A|}$. By (4.16) and (4.20), $H = F$ on $\Lambda_B(\epsilon/c_1 \alpha^{|A|}) = \Lambda_B(\epsilon_0)$. On the other hand, since F vanishes on $\Sigma_{B,0} \setminus \Lambda_B(\epsilon_0)$, by (4.20), H also vanishes on this set. This completes the proof of the last assertion of the lemma. \square

5. ABSORPTION AT THE BOUNDARY

In this section, we prove Theorem 2.3 which states that any solution of the \mathfrak{L} -martingale problem is absorbed at the boundary. Throughout this section, \mathbb{P}_x denotes a solution of the \mathfrak{L} -martingale problem starting at $x \in \Sigma$, and p is a real number satisfying

$$1 < p < b < p + 1 \quad (5.1)$$

This is possible because we assumed $b > 1$.

5.1. First time interval. Recall the definition of the hitting time σ_1 introduced in (2.10). As a first step in the proof of Theorem 2.3, we show that before σ_1 the empty sites remain empty.

Proposition 5.1. *Fix $z \in \Sigma$, and let $A = \mathcal{A}(z)$, $B = \mathcal{B}(z)$. Assume that A is nonempty. Then,*

$$\mathbb{P}_z [\|X_t\|_A = 0, 0 \leq t < \sigma_1] = 1.$$

The proof of Proposition 5.1 is divided in several steps. Fix $z \in \Sigma$, and let $A = \mathcal{A}(z)$ and $B = \mathcal{B}(z)$. Obviously, $z \in \mathring{\Sigma}_{B,0}$. Consider the function $f_A : \Sigma \rightarrow \mathbb{R}$ given by

$$f_A(x) = \pi_A(x)^{p+1}. \quad (5.2)$$

As $p > 1$, it is clear that f_A belongs to \mathcal{D}_A . We start showing that $\mathfrak{L}f_A$ is negative in a neighborhood of z . Denote by $\Sigma_B(\epsilon)$, $\epsilon > 0$, the compact neighborhood of $\Sigma_{B,0}$ defined by:

$$\Sigma_B(\epsilon) := \{x \in \Sigma : \max_{j \in A} x_j \leq \epsilon\}.$$

Recall from Section 3.E that we denote by \mathcal{L}^* the adjoint of the generator \mathcal{L} in $L^2(\mathbf{m})$, and by \mathcal{S} the symmetric part.

Lemma 5.2. *There exists $a_0 > 0$ such that for all $\epsilon > 0$ and $x \in \Sigma_B(a_0\epsilon) \cap \Lambda_B(\epsilon)$ we have $\mathfrak{L}f_A(x) \leq 0$.*

Proof. Let a_0 be given by

$$a_0^{-1} := \max_{k \in A} \frac{b \langle \mathcal{L}e_k, \mathbf{1}\{B\} \rangle_{\mathbf{m}}}{(b-p) \langle (-\mathcal{S})e_k, e_k \rangle_{\mathbf{m}}}.$$

Note that the numerator is non-negative for each $k \in A$ because $(\mathcal{L}e_k)(j) = r(j, k) \geq 0$ for $j \in B = A^c$. Furthermore, by irreducibility, $\langle \mathcal{L}e_k, \mathbf{1}\{B\} \rangle_{\mathbf{m}} = \sum_{j \in B} m(j)r(j, k)$ is strictly positive at least for one $k \in A$. This proves that $a_0^{-1} > 0$ because $b > p$.

Fix $\epsilon > 0$ and $x \in \Sigma_B(a_0\epsilon) \cap \Lambda_B(\epsilon)$. A straightforward calculation yields that for any $j \in S$,

$$(\mathbf{v}_j \cdot \nabla f_A)(x) = (p+1) \sum_{k \in A} \frac{f_A(x)}{x_k} \mathcal{L}e_k(j),$$

so that $\mathbf{b}(x) \cdot \nabla f_A(x)$ can be written as

$$b(p+1) \sum_{j,k \in A} \frac{f_A(x)}{x_j x_k} \langle \mathcal{S}e_k, e_j \rangle_{\mathbf{m}} + b(p+1) \sum_{k \in A, j \in B} \frac{f_A(x)}{x_j x_k} \langle \mathcal{L}e_k, e_j \rangle_{\mathbf{m}}.$$

In the above formula, the indicator $\mathbf{1}\{x_j > 0\}$, $j \in A$ (resp. $j \in B$), has been removed because $f_A(x)/x_j \rightarrow 0$ as $x_j \rightarrow 0$ (resp. x belongs to $\Lambda_B(\epsilon)$). On the other hand, a computation, similar to the one carried out to obtain (3.12), shows that the second-order piece of $\mathfrak{L}f_A(x)$ can be written as

$$\begin{aligned} & \sum_{j,k \in S} \partial_{x_j, x_k}^2 f_A(x) \langle (-\mathcal{S})e_j, e_k \rangle_{\mathbf{m}} \\ &= -(p+1)^2 \sum_{\substack{j,k \in A \\ j \neq k}} \frac{f_A(x)}{x_j x_k} \langle \mathcal{S}e_j, e_k \rangle_{\mathbf{m}} + p(p+1) \sum_{k \in A} \frac{f_A(x)}{x_k^2} \langle (-\mathcal{S})e_k, e_k \rangle_{\mathbf{m}}. \end{aligned}$$

It follows from the previous calculations that $(p+1)^{-1} \mathfrak{L}f_A(x)$ is equal to

$$\begin{aligned} & -(p+1-b) \sum_{\substack{j,k \in A \\ k \neq j}} \frac{f_A(x)}{x_j x_k} \langle \mathcal{S}e_j, e_k \rangle_{\mathbf{m}} + b \sum_{k \in A, j \in B} \frac{f_A(x)}{x_k x_j} \langle \mathcal{L}e_k, e_j \rangle_{\mathbf{m}} \\ & - (b-p) \sum_{k \in A} \frac{f_A(x)}{x_k^2} \langle (-\mathcal{S})e_k, e_k \rangle_{\mathbf{m}}. \end{aligned} \quad (5.3)$$

Since $p+1-b > 0$ the first term in the above expression is clearly negative. As $x \in \Lambda_B(\epsilon)$, the second term is bounded above by

$$\frac{b}{\epsilon} \sum_{k \in A} \frac{f_A(x)}{x_k} \langle \mathcal{L}e_k, \mathbf{1}\{B\} \rangle_{\mathbf{m}}.$$

Since $b > p$ and $x \in \Sigma_B(a_0\epsilon)$ the last term is bounded above by

$$-\frac{(b-p)}{a_0\epsilon} \sum_{k \in A} \frac{f_A(x)}{x_k} \langle (-\mathcal{S})e_k, e_k \rangle_{\mathbf{m}}.$$

By the last two estimates and by definition of a_0 we conclude that the expression in (5.3) is negative. \square

In virtue of Lemma 4.4, for each $\epsilon > 0$, there exists a function in \mathcal{D}_S , denoted by H_A^ϵ , such that

$$H_A^\epsilon(x) = f_A(x) \quad \text{and} \quad \mathcal{L}H_A^\epsilon(x) = \mathcal{L}f_A(x), \quad x \in \Lambda_B(\epsilon). \quad (5.4)$$

For every $\epsilon > 0$, denote by $\tau_0(\epsilon)$ the exit time from the compact set $\Sigma_B(a_0\epsilon) \cap \Lambda_B(\epsilon)$:

$$\tau_0(\epsilon) := \inf\{t \geq 0 : X_t \notin \Sigma_B(a_0\epsilon) \cap \Lambda_B(\epsilon)\}.$$

Lemma 5.3. *For every $0 < \epsilon < \min_{j \in B} z_j$ we have*

$$\mathbb{P}_z[\pi_A(X_t) = 0, \quad 0 \leq t \leq \tau_0(\epsilon)] = 1.$$

Proof. Fix $t > 0$. Since $H_A^\epsilon \in \mathcal{D}_S$ we have

$$\mathbb{E}_z[H_A^\epsilon(X_{t \wedge \tau_0(\epsilon)})] = H_A^\epsilon(z) + \mathbb{E}_z\left[\int_0^{t \wedge \tau_0(\epsilon)} \mathcal{L}H_A^\epsilon(X_s) ds\right].$$

By definition of $\tau_0(\epsilon)$, by (5.4) and by Lemma 5.2, the expectation on the right hand side in the above equation is negative. Hence

$$\mathbb{E}_z[H_A^\epsilon(X_{t \wedge \tau_0(\epsilon)})] \leq H_A^\epsilon(z).$$

By (5.4) and since $\epsilon < \min_{j \in B} z_j$ we may replace H_A^ϵ by f_A in the above inequality. Since $f_A(z) = 0$, after this replacement we have that $\mathbb{E}_z[f_A(X_{t \wedge \tau_0(\epsilon)})] \leq 0$. This proves that for all $t \geq 0$,

$$\mathbb{P}_z[\pi_A(X_{t \wedge \tau_0(\epsilon)}) = 0] = 1.$$

To complete the proof it remains to consider a countable set of times dense in \mathbb{R}_+ . \square

We have thus shown that, under \mathbb{P}_z , before time $\tau_0(\epsilon)$ at least one of the coordinates in A must be zero. To improve this result, we consider for each nonempty subset $D \subseteq A$ the function $f_D : \Sigma_{D \cup B} \rightarrow \mathbb{R}$ defined as $f_D(x) = \prod_{j \in D} x_j^{p+1}$ so that the definition of f_A is consistent with (5.2).

At this point, we reduce the neighborhood of z to obtain estimates, similar to the ones derived in Lemma 5.2, for all functions f_D . For each nonempty $D \subseteq A$, let $a_D > 0$ be given by

$$a_D^{-1} = \max_{k \in A} \frac{b \langle \mathcal{L}^{B \cup D} e_k, \mathbf{1}\{B\} \rangle_{\mathbf{m}}}{(b-p) \langle (-\mathcal{S}^{B \cup D}) e_k, e_k \rangle_{\mathbf{m}}}$$

where here $\langle \cdot, \cdot \rangle_{\mathbf{m}}$ represents the L^2 -inner product with respect to \mathbf{m} restricted to $B \cup D$, $\{e_k : k \in B \cup D\}$ the canonical basis for $\mathbb{R}^{B \cup D}$, and $\mathcal{S}^{B \cup D}$ the symmetric part of the generator $\mathcal{L}^{B \cup D}$ in $L^2(\mathbf{m})$. Since $\mathbf{r}^{B \cup D}$ is irreducible and $b > p$, a_D is well defined and strictly positive for all nonempty subsets D of A .

We set $\mathbf{a}_0 := \min\{a_D : \emptyset \subsetneq D \subseteq A\}$ and denote

$$K_z(\epsilon) := \Sigma_B(\mathbf{a}_0\epsilon) \cap \Lambda_B(\epsilon),$$

for all $\epsilon > 0$. Of course, $z \in K_z(\epsilon)$ if and only if $\epsilon \in (0, \epsilon_0)$ where

$$\epsilon_0 := \min_{j \in B} z_j > 0.$$

Lemma 5.4. *Let D be a nonempty subset of A . For all $\epsilon \in (0, \epsilon_0)$,*

$$\mathfrak{L}_{D \cup B} f_D(x_{D \cup B}) \leq 0, \quad x \in K_z(\epsilon).$$

Proof. The proof is similar to the one of Lemma 5.2. One just needs to replace \mathfrak{L} , \mathcal{L} and \mathcal{S} by the respective operators $\mathfrak{L}_{B \cup D}$, $\mathcal{L}^{B \cup D}$ and $\mathcal{S}^{B \cup D}$. \square

For each nonempty proper subset D of A , Lemma 4.3 permits to extend the function $f_D : \Sigma_{D \cup B} \rightarrow \mathbb{R}$ to a function $F_D : \Sigma \rightarrow \mathbb{R}$ which belongs to \mathcal{D}_A and such that:

$$\begin{aligned} F_D(x) &= f_D(x_{B \cup D}) = \pi_D(x)^{p+1}, \quad x \in \Sigma_{B \cup D, 0}, \\ \mathfrak{L}F_D(x) &= \mathfrak{L}_{B \cup D} f_D(x_{B \cup D}), \quad x \in \mathring{\Sigma}_{B \cup D, 0}, \\ \mathfrak{L}F_D(x) &= 0 \quad \text{if } \pi_D(x) = 0. \end{aligned} \tag{5.5}$$

Moreover, since each f_D is positive, it follows from the construction presented in Lemma 4.3 that $F_D(x) \geq 0$ for all $x \in \Sigma$. Denote by H_D^ϵ the function in \mathcal{D}_S obtained by Lemma 4.4 from F_D . Thus, for each $\epsilon \in (0, \epsilon_0)$ and for each nonempty, proper subset D of A ,

$$H_D^\epsilon(x) = F_D(x) \quad \text{and} \quad \mathfrak{L}H_D^\epsilon(x) = \mathfrak{L}F_D(x), \quad x \in \Lambda_B(\epsilon). \tag{5.6}$$

Since $z \in K_z(\epsilon) \cap \Sigma_{D \cup B, 0}$, by the first property in (5.5), by (5.6), and by the positivity of F_D ,

$$\begin{aligned} H_D^\epsilon(z) &= F_D(z) = \pi_D(z)^{p+1} = 0, \\ H_D^\epsilon(x) &= F_D(x) \geq 0, \quad x \in \Lambda_B(\epsilon), \\ H_D^\epsilon(x) &= F_D(x) = \pi_D(x)^{p+1}, \quad x \in \Lambda_B(\epsilon) \cap \Sigma_{B \cup D, 0}. \end{aligned} \tag{5.7}$$

Lemma 5.5. *For all $\epsilon > 0$ there exists a constant $C(\epsilon) > 0$ such that for all nonempty, proper subset D of A ,*

$$\mathfrak{L}H_D^\epsilon(x) \leq C(\epsilon) \mathbf{1}\{\pi_D(x) > 0, \|x\|_{A \setminus D} > 0\}, \quad x \in K_z(\epsilon).$$

Proof. Fix $\epsilon > 0$. Since each function $\mathfrak{L}H_D^\epsilon, \emptyset \subsetneq D \subsetneq A$, is continuous on Σ ,

$$C(\epsilon) := \sup \{|\mathfrak{L}H_D^\epsilon(x)| : x \in \Sigma, \emptyset \subsetneq D \subsetneq A\} < \infty.$$

Fix a nonempty subset D of A and $x \in K_z(\epsilon)$. Since $x \in \Lambda_B(\epsilon)$, by (5.6) and by the third property in (5.5),

$$\mathfrak{L}H_D^\epsilon(x) = \mathfrak{L}H_D^\epsilon(x) \mathbf{1}\{\pi_D(x) > 0\}.$$

On the other hand, if $\pi_D(x) > 0$ and $\|x\|_{A \setminus D} = 0$, $x \in \mathring{\Sigma}_{D \cup B, 0}$. Hence, in this case, by (5.6) and by the second property in (5.5),

$$\mathfrak{L}H_D^\epsilon(x) = \mathfrak{L}F_D(x) = \mathfrak{L}_{D \cup B} f_D(x_{D \cup B}).$$

From this observation and from Lemma 5.4 we conclude that

$$\mathfrak{L}H_D^\epsilon(x) \leq 0 \quad \text{if } \pi_D(x) > 0 \quad \text{and} \quad \|x\|_{A \setminus D} = 0.$$

It follows from the previous estimates that

$$\mathfrak{L}H_D^\epsilon(x) \leq \mathbf{1}\{\pi_D(x) > 0, \|x\|_{A \setminus D} > 0\} \mathfrak{L}H_D^\epsilon(x).$$

The assertion of the lemma is a simple consequence of this inequality and the definition of $C(\epsilon)$. \square

We may now improve Lemma 5.3 in the following sense. For every $\epsilon \in (0, \epsilon_0)$, define $\tau(\epsilon)$ as the exit time from the compact neighborhood $K_z(\epsilon)$ of z :

$$\tau(\epsilon) := \inf\{t \geq 0 : X_t \notin K_z(\epsilon)\}.$$

Lemma 5.6. *For all $\epsilon \in (0, \epsilon_0)$ and nonempty subset D of A ,*

$$\mathbb{P}_z[\pi_D(X_t) = 0, 0 \leq t \leq \tau(\epsilon)] = 1.$$

Proof. Fix $\epsilon \in (0, \epsilon_0)$. By Lemma 5.3, the claim holds for $D = A$. We extend the assertion to all nonempty $D \subseteq A$ by a recursive argument. Fix $0 \leq n < |A| - 1$, and assume that, the assertion of the lemma holds for all $D \subseteq A$ with $|D| \geq |A| - n$. Consider a subset $D' \subseteq A$ such that $|D'| = |A| - n - 1$. By the recurrence hypothesis,

$$\mathbb{P}_z[\pi_{D'}(X_{s \wedge \tau(\epsilon)}) > 0, \|X_{s \wedge \tau(\epsilon)}\|_{A \setminus D'} > 0] = 0, \quad (5.8)$$

for all $s \geq 0$. Fix $t \geq 0$. Since $H_{D'}^\epsilon \in \mathcal{D}_S$,

$$\mathbb{E}_z[H_{D'}^\epsilon(X_{t \wedge \tau(\epsilon)})] = H_{D'}^\epsilon(z) + \mathbb{E}_z\left[\int_0^{t \wedge \tau(\epsilon)} \mathfrak{L}H_{D'}^\epsilon(X_s) ds\right].$$

Thus, by the first property in (5.7) and by Lemma 5.5, we get

$$\begin{aligned} & \mathbb{E}_z[H_{D'}^\epsilon(X_{t \wedge \tau(\epsilon)})] \\ & \leq C(\epsilon) \mathbb{E}_z\left[\int_0^{t \wedge \tau(\epsilon)} \mathbf{1}_{\{\pi_{D'}(X_s) > 0, \|X_s\|_{A \setminus D'} > 0\}} ds\right]. \end{aligned}$$

By (5.8), the right hand side of the previous expression vanishes. By the last property in (5.7), on the set $\{\|X_{t \wedge \tau(\epsilon)}\|_{A \setminus D'} = 0\}$, $H_{D'}^\epsilon(X_{t \wedge \tau(\epsilon)}) = \pi_{D'}(X_{t \wedge \tau(\epsilon)})^{p+1}$. Therefore, by the second property of (5.7),

$$\mathbb{E}_z[\mathbf{1}_{\{\|X_{t \wedge \tau(\epsilon)}\|_{A \setminus D'} = 0\}} \pi_{D'}(X_{t \wedge \tau(\epsilon)})^{p+1}] \leq \mathbb{E}_z[H_{D'}^\epsilon(X_{t \wedge \tau(\epsilon)})] = 0,$$

so that

$$\mathbb{P}_z[\|X_{t \wedge \tau(\epsilon)}\|_{A \setminus D'} = 0, \pi_{D'}(X_{t \wedge \tau(\epsilon)}) > 0] = 0.$$

The previous identity and (5.8) yield that

$$\mathbb{P}_z[\pi_{D'}(X_{t \wedge \tau(\epsilon)}) > 0] = 0.$$

Finally, taking a countable set of times t , dense in \mathbb{R}_+ , we conclude that the assertion of the lemma holds for D' . This completes the proof. \square

The previous lemma with $D = \{j\}$, $j \in A$, yields that, for any $\epsilon \in (0, \epsilon_0)$,

$$\mathbb{P}_z[\|X_t\|_A = 0 \text{ for all } 0 \leq t \leq \tau(\epsilon)] = 1.$$

Since $\tau(\epsilon)$ is the first time t in which either $\max_{j \in A} X_t(j) > \alpha_0 \epsilon$ or $\min_{j \in B} X_t(j) < \epsilon$,

$$\mathbb{P}_z[\|X_t\|_A = 0 \text{ for all } 0 \leq t \leq h_B(\epsilon)] = 1,$$

where, $h_B(\epsilon)$ is the exit time of $\Lambda_B(\epsilon)$:

$$h_B(\epsilon) := \inf\{t \geq 0 : \min_{j \in B} X_t(j) < \epsilon\}, \quad \epsilon > 0. \quad (5.9)$$

To complete the proof of Proposition 5.1, it remains to let $\epsilon \downarrow 0$.

5.2. Absorption at the boundary. We prove that \mathbb{P}_x is absorbing for every $x \in \Sigma$. This result follows from next proposition.

Proposition 5.7. *For all $x \in \Sigma$, $n \geq 0$,*

$$\mathbb{P}_x[\sigma_n = \infty \text{ or } \mathcal{A}_n = \mathcal{A}(X_t) \text{ for all } t \in [\sigma_n, \sigma_{n+1})] = 1.$$

The assertion for $n = 0$ has been proved in Proposition 5.1. To extend this claim to the remaining time intervals we use the concept of regular conditional probability distributions (r.c.p.d.) and the techniques introduced in [16]. Given a probability measure \mathbb{P} on $C(\mathbb{R}_+, \Sigma)$ and $n \geq 1$, for $\omega \in \{\sigma_n < \infty\}$, define a set of probability measures \mathbb{P}_ω^n on $C(\mathbb{R}_+, \Sigma)$ as follows. First, choose a r.c.p.d. $\{\mathbb{P}_\omega\}$ for \mathbb{P} given the σ -field \mathcal{F}_{σ_n} . Then, define

$$\mathbb{P}_\omega^n := \mathbb{P}_\omega \circ \theta_{\sigma_n(\omega)}^{-1}, \quad \text{for } \omega \in \{\sigma_n < \infty\}, \quad (5.10)$$

where we recall that $(\theta_t)_{t \geq 0}$ stands for the semigroup of time translations. Next lemma is an immediate consequence of Theorem 1.2.10 in [16].

Lemma 5.8. *Let \mathbb{P} be a solution of the \mathcal{L} -martingale problem and $n \geq 1$. Given $H \in \mathcal{D}_S$ there exists a \mathbb{P} -null set $\mathcal{N} \in \mathcal{F}_{\sigma_n}$ such that, for all $\omega \in \mathcal{N}^c \cap \{\sigma_n < \infty\}$,*

$$H(X_t) - \int_0^t \mathcal{L}H(X_s) ds, \quad t \geq 0$$

is a \mathbb{P}_ω^n -martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$.

This lemma permits to employ the arguments presented in the proof of Proposition 5.1 to the general setting of Proposition 5.7.

Proof of Proposition 5.7. Fix $x \in \Sigma$ and $n \geq 1$. To keep notation simple denote by \mathbb{P}_ω^n the measure $(\mathbb{P}_x)_\omega^n$ defined by (5.10). By taking the conditional expectation with respect to \mathcal{F}_{σ_n} in the probability appearing in the statement of Proposition 5.7, we conclude that it is enough to show that

$$\mathbb{P}_\omega^n\{\mathcal{A}_n(\omega) = \mathcal{A}(X_t), 0 \leq t < \sigma_1\} = 1, \quad (5.11)$$

for \mathbb{P}_x -almost all $\omega \in \{\sigma_n < \infty\}$. By Lemma 5.8, there exists a \mathbb{P}_x -null set $\mathcal{N} \in \mathcal{F}_{\sigma_n}$ such that, for all $\omega \in \mathcal{N}^c \cap \{\sigma_n < \infty\}$, for all nonempty subsets D of S and for a sequence $\epsilon_k \downarrow 0$,

$$H_D^{\epsilon_k}(X_t) - \int_0^t \mathcal{L}H_D^{\epsilon_k}(X_s) ds, \quad t \geq 0$$

is a \mathbb{P}_ω^n -martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$, where H_D^ϵ are the functions introduced in the previous subsection. At this point, we may repeat the argument presented in the proof of Proposition 5.1 to conclude that (5.11) holds for all $\omega \in \mathcal{N}^c \cap \{\sigma_n < \infty\}$. \square

Theorem 2.3 is a simple consequence of Proposition 5.7.

6. UNIQUENESS

In this section, we prove that for any $x \in \Sigma$ there exists at most one solution of the \mathcal{L} -martingale problem starting at x . We start showing that any such solution also solves the martingale problem determined by \mathcal{L} in the form stated in Theorem 2.5. Then, we show in Proposition 6.1 that this fact along with the absorbing property, proved in the previous section, provides the desired uniqueness.

Proof of Theorem 2.5. Fix $z \in \Sigma$ and a function $F \in D_0(\Sigma)$. Let

$$M_t^F := F(X_t) - \int_0^t \mathcal{L}F(X_s)ds - \int_0^t F(X_s)dN_s^S, \quad t \geq 0.$$

Clearly $(M_t^F)_{t \geq 0}$ is \mathcal{F}_t -adapted and M_t^F is bounded for each $t \geq 0$. For each proper subset B of S , with $|B| \geq 2$, define $G_B : \Sigma \rightarrow \mathbb{R}$ as $G_B = [F]_B \circ \Upsilon$, where Υ is the linear map defined in (3.5). By Lemma 3.1, G_B belongs to \mathcal{D}_A . Since $[F]_B$ has compact support contained in $\overset{\circ}{\Sigma}_B$, we may apply the last assertion in Lemma 4.4 to each G_B . In this way, for each proper subset B of S with at least two elements, we obtain a function $H_B \in \mathcal{D}_S$ such that

$$\begin{aligned} H_B(x) &= G_B(x) = F(x) \mathbf{1}\{x \in \overset{\circ}{\Sigma}_B\}, \quad x \in \Sigma_{B,0}, \\ \mathfrak{L}H_B(x) &= \mathfrak{L}G_B(x) = \mathfrak{L}_B[F]_B(x_B), \quad x \in \Lambda_B(\epsilon) \cap \Sigma_{B,0}. \end{aligned} \quad (6.1)$$

By continuity of Υ , we may choose $\epsilon > 0$ small enough for G_B to vanish in a neighborhood of $\Sigma_{B,0} \setminus \Lambda_B(\epsilon)$. For such ϵ ,

$$\mathfrak{L}H_B(x) = \mathfrak{L}G_B(x) = 0 = \mathfrak{L}_B[F]_B(x_B), \quad x \in \Sigma_{B,0} \setminus \Lambda_B(\epsilon).$$

The identity $\mathfrak{L}H_B(x) = 0$ and $\mathfrak{L}_B[F]_B(x_B) = 0$ are in force because H_B and $[F]_B$ vanish if x_j is small enough for some $j \in B$. On the other hand, by its definition, the function G_B vanishes if $x_j + \|x\|_A$ is small enough for some $j \in B$, which explains why $\mathfrak{L}G_B(x) = 0$.

It follows from the two previous displayed equations that

$$\mathfrak{L}H_B(x) = \mathcal{L}F(x), \quad x \in \overset{\circ}{\Sigma}_{B,0}. \quad (6.2)$$

In addition, define H_S as equal to $[F]_S$ (which is equal to $F \mathbf{1}\{\overset{\circ}{\Sigma}\}$) and, for all $j \in S$, $H_{\{j\}}$ as a constant function equal to $F(e_j)$ so that $H_B \in \mathcal{D}_S$, for all nonempty subset B of S . Therefore,

$$M_t^B := H_B(X_t) - \int_0^t \mathfrak{L}H_B(X_s)ds, \quad t \geq 0, \quad (6.3)$$

is a \mathbb{P}_z -martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$ for all $\emptyset \subsetneq B \subseteq S$.

On the *absorbing event*

$$\bigcap_{n \geq 0} \{\mathcal{A}_n \subseteq \mathcal{A}(X_t) \text{ for all } t \geq \sigma_n\},$$

we have that

$$F(X_t) - F(X_0) - \int_0^t F(X_s)dN_s = F(X_t) - \sum_{n=0}^{N_t} F(X_{\sigma_n}), \quad t \geq 0.$$

For each $n \geq 0$ such that $\sigma_{n+1} < \infty$, $H_{\mathcal{B}_n}(X_{\sigma_{n+1}}) = 0$ because, as already observed, $H_B(x) = 0$ if one of the coordinates x_j , $j \in B$, vanishes. Therefore, by (6.1), on the absorbing event the right hand side of the previous expression is equal to

$$\begin{aligned} & \sum_{n=0}^{N_t} \{H_{\mathcal{B}_n}(X_{\sigma_{n+1} \wedge t}) - F(X_{\sigma_n})\} \\ &= \sum_B \sum_{n=0}^{N_t} \{H_B(X_{\sigma_{n+1} \wedge t}) - H_B(X_{\sigma_n})\} \mathbf{1}\{\mathcal{B}_n = B\}, \end{aligned}$$

where the first sum on the right hand side is carried over all nonempty subsets B of S . By (6.3), for each such B , the sum

$$\sum_{n=0}^{N_t} \{H_B(X_{\sigma_{n+1} \wedge t}) - H_B(X_{\sigma_n})\} \mathbf{1}\{\mathcal{B}_n = B\},$$

can be written as

$$\begin{aligned} & \sum_{n=0}^{N_t} \left[\{M_{\sigma_{n+1} \wedge t}^B - M_{\sigma_n}^B\} + \int_{\sigma_n}^{\sigma_{n+1} \wedge t} \mathcal{L}H_B(X_s) ds \right] \mathbf{1}\{\mathcal{B}_n = B\} \\ &= \sum_{n=0}^{N_t} \{M_{\sigma_{n+1} \wedge t}^B - M_{\sigma_n}^B\} \mathbf{1}\{\mathcal{B}_n = B\} + \int_0^t \mathcal{L}H_B(X_s) \mathbf{1}\{\mathcal{B}(X_s) = B\} ds. \end{aligned}$$

By (6.2), this last expression equals

$$\widehat{M}_t^B + \int_0^t \mathcal{L}F(X_s) \mathbf{1}\{\mathcal{B}(X_s) = B\} ds$$

where

$$\widehat{M}_t^B := \sum_{n=0}^{N_t} \{M_{\sigma_{n+1} \wedge t}^B - M_{\sigma_n}^B\} \mathbf{1}\{\mathcal{B}_n = B\}, \quad t \geq 0.$$

Up to this point, we proved that

$$F(X_t) - F(X_0) - \int_0^t F(X_s) dN_s = \int_0^t \mathcal{L}F(X_s) ds + \sum_B \widehat{M}_t^B, \quad \mathbb{P}_z\text{-a. s.},$$

for all $t \geq 0$. Therefore, it remains to prove that

$$\mathbb{E}_z[\widehat{M}_t^B - \widehat{M}_s^B | \mathcal{F}_s] = 0, \quad (6.4)$$

for every $0 \leq s < t$ and nonempty subset B of S . Fix $0 \leq s < t$, $\emptyset \subsetneq B \subseteq S$ and $\mathcal{U} \in \mathcal{F}_s$. By definition,

$$\mathbb{E}_z[\{\widehat{M}_t^B - \widehat{M}_s^B\} \mathbf{1}\{\mathcal{U}\}] = \sum_{n=0}^{|S|} \mathbb{E}_z[\{M_{(\sigma_{n+1} \wedge t) \vee s}^B - M_{(\sigma_n \wedge t) \vee s}^B\} \mathbf{1}\{\mathcal{U}, \mathcal{B}_n = B\}].$$

For each $0 \leq n \leq |S|$,

$$\begin{aligned} & \mathbb{E}_z[\{M_{(\sigma_{n+1} \wedge t) \vee s}^B - M_{(\sigma_n \wedge t) \vee s}^B\} \mathbf{1}\{\mathcal{U}, \mathcal{B}_n = B\}] \\ &= \mathbb{E}_z[\{M_{(\sigma_{n+1} \wedge t) \vee s}^B - M_{(\sigma_n \wedge t) \vee s}^B\} \mathbf{1}\{\mathcal{U}, \mathcal{B}_n = B, \sigma_n \leq t\}] = 0, \end{aligned}$$

where last equality follows from the martingale property of $(M_t^B)_{t \geq 0}$ and from the fact that

$$\mathcal{U} \cap \{\mathcal{B}_n = B\} \cap \{\sigma_n \leq t\} \in \mathcal{F}_{(\sigma_n \wedge t) \vee s}.$$

This proves (6.4) and completes the proof of the theorem. \square

A probability measure \mathbb{P} on $C(\mathbb{R}_+, \Sigma)$ is said to be an absorbing solution of the \mathcal{L} -martingale problem if \mathbb{P} is absorbing and for all $F \in D_0(\Sigma)$,

$$F(X_t) - \int_0^t \mathcal{L}F(X_s) ds - \int_0^t F(X_s) dN_s^S, \quad t \geq 0,$$

is a \mathbb{P} -martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$.

Proposition 6.1. *For each $x \in \Sigma$, there exists at most one absorbing solution of the \mathcal{L} -martingale problem starting at x .*

We first prove, in Lemma 6.2 below, uniqueness for the absorbing solution on the time interval $[\sigma_0, \sigma_1)$. For each $B \subseteq S$ with at least two elements, let

$$\mathbf{b}_\epsilon^B(x) = b \sum_{j \in B} \frac{m_j}{\epsilon \vee x_j} \mathbf{v}_j^B, \quad x \in \mathbb{R}^B,$$

Thus, for every $\epsilon > 0$, $\mathbf{b}_\epsilon^B : \mathbb{R}^B \rightarrow \mathbb{R}^B$ is a bounded, continuous vector field which coincides with \mathbf{b}^B on

$$\Lambda_{B,\epsilon} := \{x \in \Sigma_B : \min_{j \in B} x_j \geq \epsilon\},$$

Let \mathbf{a}^B be the matrix whose entries $(\mathbf{a}^B(j, k))_{j, k \in B}$ are given by

$$\mathbf{a}_{j,k}^B := \langle \mathbf{e}_j, -\mathcal{L}^B \mathbf{e}_k \rangle_{\mathbf{m}}, \quad j, k \in B,$$

where, by abuse of notation, $\{\mathbf{e}_j : j \in B\}$ and $\langle \cdot, \cdot \rangle_{\mathbf{m}}$ represent the canonical basis of \mathbb{R}^B and the scalar product with respect to \mathbf{m} restricted to B , respectively. Consider the symmetric matrix $\mathbf{a}_s^B = (1/2)(\mathbf{a}^B + (\mathbf{a}^B)^t)$ so that,

$$\sum_{j, k \in B} v_j \mathbf{a}_s^B(j, k) w_k = \langle v, (-\mathcal{S}^B)w \rangle_{\mathbf{m}}, \quad v, w \in \mathbb{R}^B.$$

Note that for all $\epsilon > 0$ and $F \in C^2(\Sigma_B)$,

$$\mathfrak{L}_B F(x) = \mathbf{b}_\epsilon^B(x) \cdot \nabla F(x) + \text{Tr}[\mathbf{a}_s^B \times \text{Hess } F(x)], \quad x \in \Lambda_{B,\epsilon}. \quad (6.5)$$

Let $(X_t^B)_{t \geq 0}$ be the coordinate maps in the path space $C(\mathbb{R}_+, \mathbb{R}^B)$, let $\mathcal{F}_t^B := \sigma(X_s : 0 \leq s \leq t)$, $t \geq 0$, and denote by $h_{B,\epsilon}$ the exit time from $\Lambda_{B,\epsilon}$:

$$h_{B,\epsilon} := \inf\{t \geq 0 : X_t^B \notin \Lambda_{B,\epsilon}\}, \quad \epsilon > 0.$$

Denote by $\mathbb{Q}_x^{B,\epsilon}$, $x \in \mathbb{R}^B$, $\epsilon > 0$, the unique solution of the $(\mathbf{b}_\epsilon^B, \mathbf{a}^B)$ -martingale problem starting at x . Namely, $\mathbb{Q}_x^{B,\epsilon}$ is the unique probability measure on $C(\mathbb{R}_+, \mathbb{R}^B)$ such that $\mathbb{Q}_x^{B,\epsilon}\{X_0^B = x\} = 1$ and such that for all $H : \mathbb{R}^B \rightarrow \mathbb{R}$ of class C^2 and of compact support,

$$H(X_t^B) - \int_0^t \left\{ \mathbf{b}_\epsilon^B(X_s^B) \cdot \nabla H(X_s^B) + \text{Tr}[\mathbf{a}_s^B \times \text{Hess } H(X_s^B)] \right\} ds, \quad t \geq 0$$

is a $\mathbb{Q}_x^{B,\epsilon}$ -martingale with respect to $(\mathcal{F}_t^B)_{t \geq 0}$.

Since \mathbf{r}^B is irreducible, for all v in $\mathbb{R}^B \setminus \{0\}$,

$$\sum_{j, k \in B} v_j \mathbf{a}_s^B(j, k) v_k = \langle v, (-\mathcal{S}^B)v \rangle_{\mathbf{m}} > 0.$$

Uniqueness of $\{\mathbb{Q}_x^{B,\epsilon}\}$ follows from Theorem 7.1.9 in [16] and from the previous equation. It also follows from this theorem that $\{\mathbb{Q}_x^{B,\epsilon} : x \in \mathbb{R}^B\}$ is Feller continuous.

The next result asserts that, in the time interval $[0, h_{B,\epsilon})$, an absorbing solution of the \mathcal{L} -martingale problem starting at x coincides with $\mathbb{Q}_x^{B,\epsilon}$.

Lemma 6.2. *Fix $x \in \Sigma$, and let $B = \mathcal{B}(x)$. Denote by \mathbb{P}_x an absorbing solution of the \mathcal{L} -martingale problem starting at x , and by \mathbb{P}_x^B the law on $C(\mathbb{R}_+, \mathbb{R}^B)$ of the path*

$$(X_t(j), j \in B), \quad t \geq 0$$

under \mathbb{P}_x . Then, for all $\epsilon > 0$, $\mathbb{P}_x^B \equiv \mathbb{Q}_x^{B,\epsilon}$ on $\mathcal{F}_{h_{B,\epsilon}}^B$. In particular, if \mathbb{P}_1 and \mathbb{P}_2 are two absorbing solutions of the \mathcal{L} -martingale problem starting at x , then $\mathbb{P}_1 \equiv \mathbb{P}_2$ on \mathcal{F}_{σ_1} .

Proof. Fix a starting point $z \in \Sigma$ and let $B = \mathcal{B}(z)$. Denote by \mathbb{P}_z an absorbing solution of the \mathcal{L} -martingale problem starting at z . Fix $\epsilon > 0$, and $H : \mathbb{R}^B \rightarrow \mathbb{R}$ of class C^2 and of compact support. Let

$$M_t^{H,\epsilon} := H(X_t^B) - \int_0^t \left\{ \mathbf{b}_\epsilon^B(X_s^B) \cdot \nabla H(X_s^B) + \text{Tr}(\mathbf{a}_s^B \times \text{Hess } H(X_s^B)) \right\} ds, \quad t \geq 0.$$

It is easy to obtain a function $F \in D_0(\Sigma)$ such that

$$F(x) = H(x_B) \quad \text{for all } x \in \Lambda_B(\epsilon/2) \cap \Sigma_{B,0}.$$

Recall the definition of $h_B(\epsilon)$ introduced in (5.9). By assumption,

$$F(X_{t \wedge h_B(\epsilon)}) - \int_0^{t \wedge h_B(\epsilon)} \mathcal{L}F(X_s) ds, \quad t \geq 0,$$

is a \mathbb{P}_z -martingale with respect to $(\mathcal{F}_t)_{t \geq 0}$. By definition of $h_B(\epsilon)$ and \mathcal{L} , the last two identities yield that

$$H(X_{t \wedge h_{B,\epsilon}}^B) - \int_0^{t \wedge h_{B,\epsilon}} \mathcal{L}_B H(X_s^B) ds, \quad t \geq 0,$$

is a \mathbb{P}_z^B -martingale with respect to $(\mathcal{F}_t^B)_{t \geq 0}$. Therefore, by (6.5) and by definition of $h_{B,\epsilon}$,

$$(M_{t \wedge h_{B,\epsilon}}^{H,\epsilon})_{t \geq 0} \text{ is a } \mathbb{P}_z^B\text{-martingale with respect to } (\mathcal{F}_t^B)_{t \geq 0}. \quad (6.6)$$

We now join \mathbb{P}_z^B and $\{\mathbb{Q}_z^{B,\epsilon}\}$ at time $h_{B,\epsilon}$ as follows. To keep notation simple let $h := h_{B,\epsilon}$ and let

$$X_h^B(\omega) := X_{h(\omega)}^B(\omega), \quad \omega \in \{h_{B,\epsilon} < \infty\}.$$

Since $\{\mathbb{Q}_x^{B,\epsilon}\}$ is Feller continuous, it follows from Theorem 6.1.2 in [16] that there exists a probability \mathbb{Q} on $C(\mathbb{R}_+, \mathbb{R}^B)$ satisfying the following two properties:

- (i) \mathbb{Q} coincides with \mathbb{P}_z^B on \mathcal{F}_h^B .
- (ii) For any $\{\mathbb{Q}_\omega\}$, a r.c.p.d for \mathbb{Q} given \mathcal{F}_h^B , there exists a \mathbb{Q} -null set $\mathcal{N} \in \mathcal{F}_h^B$ such that

$$\mathbb{Q}_\omega \circ \theta_{h(\omega)}^{-1} = \mathbb{Q}_{X_h^B(\omega)}^{B,\epsilon}, \quad \text{for all } \omega \in \mathcal{N}^c \cap \{h < \infty\}.$$

Note that in Theorem 6.1.2 of [16], h is assumed to be finite. Nevertheless, the proof of this theorem can easily be adapted for a general stopping time.

By definition of $\{\mathbb{Q}_x^{B,\epsilon}\}$ and by (ii), the process $(M_t^{H,\epsilon})_{t \geq 0}$ is a $\mathbb{Q}_\omega \circ \theta_{h(\omega)}^{-1}$ -martingale, with respect to $(\mathcal{F}_t^B)_{t \geq 0}$, for all $\omega \in \mathcal{N}^c \cap \{h < \infty\}$. By Theorem 1.2.10 in [16], this fact along with (6.6) and item (i) above allows us to conclude that $(M_t^{H,\epsilon})_{t \geq 0}$ is a \mathbb{Q} -martingale. We have thus proved that \mathbb{Q} is a solution of the $(\mathbf{b}_\epsilon^B, \mathbf{a}_s^B)$ -martingale problem. Since $\mathbb{Q}\{X_0^B = z_B\} = 1$, by uniqueness, $\mathbb{Q} = \mathbb{Q}_{z_B}^{B,\epsilon}$. This fact and item (i) completes the proof of the first assertion of the lemma.

The second assertion follows from the absorbing property and from the first assertion by letting $\epsilon \downarrow 0$. \square

Given a probability measure \mathbb{P} on $C(\mathbb{R}_+, \Sigma)$, recall from (5.10) the definition of the measure $\mathbb{P}_\omega^n, \omega \in \{\sigma_n < \infty\}, n \geq 1$. We use the probability measures $\mathbb{P}_\omega^1, \omega \in \{\sigma_1 < \infty\}$, to conclude the proof of the uniqueness stated in Proposition 6.1. The proof of next lemma follows the same argument as in the proof of Theorem 6.1.3 in [16].

Lemma 6.3. *Let \mathbb{P} be an absorbing solution of the \mathcal{L} -martingale problem. Then, there exists a \mathbb{P} -null set $\mathcal{N} \in \mathcal{F}_{\sigma_1}$ such that, for all $\omega \in \mathcal{N}^c \cap \{\sigma_1 < \infty\}$, \mathbb{P}_ω^1 is an absorbing solution of the \mathcal{L} -martingale problem starting at $X_{\sigma_1}(\omega)$.*

Proof. Let Θ be a countable subset of $D_0(\Sigma)$ satisfying the following property: for all $F \in D_0(\Sigma)$, there exists a sequence $(F_n)_{n \geq 1}$ in Θ such that

$$\lim_{n \rightarrow \infty} \sup_{x \in \Sigma} \{ |F_n(x) - F(x)| + |\mathcal{L}F_n(x) - \mathcal{L}F(x)| \} = 0.$$

By Theorem 1.2.10 of [16], there exists a \mathbb{P} -null set $\mathcal{N} \in \mathcal{F}_{\sigma_1}$ such that, for all $\omega \in \mathcal{N}^c \cap \{\sigma_1 < \infty\}$ and for all $F \in \Theta$,

$$F(X_t) - \int_0^t \mathcal{L}F(X_s) ds - \int_0^t F(X_s) dN_s^S, \quad t \geq 0,$$

is a \mathbb{P}_ω^1 -martingale. Approximating a function F in $D_0(\Sigma)$ by a sequence in Θ , we may conclude that the previous expression is also a \mathbb{P}_ω^1 -martingale for all $F \in D_0(\Sigma)$.

Finally, it is easy to see that the \mathbb{P} -null set $\mathcal{N} \in \mathcal{F}_{\sigma_1}$ may be chosen so that \mathbb{P}_ω^1 is absorbing for all $\omega \in \mathcal{N}^c \cap \{\sigma_1 < \infty\}$. \square

We are now in position to complete the proof of uniqueness of absorbing solutions of the \mathcal{L} -martingale problem.

Proof of Proposition 6.1. If the starting point belongs to $\{e_j : j \in S\}$, then the claim is simple consequence of the absorbing property.

We proceed by induction. Suppose that the claim is valid for any starting point in $\{x \in \Sigma : |\mathcal{B}(x)| \leq n\}$ for some $1 \leq n < |S|$. Fix some $z \in \Sigma$ such that $|\mathcal{B}(z)| = n + 1$ and let $\mathbb{P}_i, i = 1, 2$, be two absorbing solutions of the \mathcal{L} -martingale problem starting at z . By Lemma 6.2,

$$\mathbb{P}_1 \equiv \mathbb{P}_2 \quad \text{on} \quad \mathcal{F}_{\sigma_1}. \quad (6.7)$$

From the absorbing property it follows that

$$\mathbb{P}_i[\sigma_1 < \infty \text{ and } |\mathcal{B}_1| > n] = 0, \quad i = 1, 2.$$

By Lemma 6.3, for $i = 1, 2$, there exists a \mathbb{P}_i -null set $\mathcal{N}_i \in \mathcal{F}_{\sigma_1}$ such that, for all $\omega \in \mathcal{N}_i^c \cap \{\sigma_1 < \infty\}$, $(\mathbb{P}_i)_\omega^\sigma$ is an absorbing solution of the \mathcal{L} -martingale problem starting at $X_{\sigma_1}(\omega)$. Take

$$\mathcal{N} := \mathcal{N}_1 \cup \mathcal{N}_2 \cup \{\sigma_1 < \infty \text{ and } |\mathcal{B}_1| > n\}.$$

It follows from the previous displayed equations that $\mathbb{P}_2(\mathcal{N}) = \mathbb{P}_1(\mathcal{N}) = 0$. Fix an arbitrary $\omega \in \mathcal{N}^c \cap \{\sigma_1 < \infty\}$. On the one hand, $(\mathbb{P}_1)_\omega^\sigma$ and $(\mathbb{P}_2)_\omega^\sigma$ are absorbing solution of the \mathcal{L} -martingale problem starting at $X_{\sigma_1}(\omega)$. On the other hand, by definition of \mathcal{N} , $X_{\sigma_1}(\omega)$ belongs to $\{x \in \Sigma : |\mathcal{B}(x)| \leq n\}$. Hence, by the inductive hypothesis, $(\mathbb{P}_1)_\omega^\sigma = (\mathbb{P}_2)_\omega^\sigma$. The assertion of the proposition follows from this fact and from (6.7). \square

Proof of Theorem 2.2. Theorem 2.2 follows from Theorem 2.5 and Proposition 6.1. \square

Proof of Proposition 2.4. Fix x in Σ and assume that $\mathcal{A}(x) = \{j \in S : x_j = 0\} \neq \emptyset$. Let $B = \mathcal{A}(x)^c$. It is clear that the measure \mathbb{P}_x^B starts at x_B and that it is absorbing. By the proof of Theorem 2.5, it solves the \mathcal{L} -martingale problem restricted to Σ_B : for all functions $F \in D_0(\Sigma_B)$,

$$M_t^F := F(X_t) - \int_0^t \mathcal{L}F(X_s) ds - \int_0^t F(X_s) dN_s^S, \quad t \geq 0.$$

is a \mathbb{P}_x^B -martingale. The assertion of the proposition follows now from the uniqueness stated in Proposition 6.1. \square

7. EXISTENCE

In this section we prove the convergence stated in Theorem 2.6. This result also guarantees the existence of solutions of the \mathfrak{L} -martingale problem. By abuse of notation, in this section, we also denote by $(X_t)_{t \geq 0}$ the coordinate process defined on $D(\mathbb{R}_+, \Sigma)$.

7.1. Tightness. We prove in this section that, for any sequence $x_N \in \Sigma_N$, $N \geq 1$, the sequence of probability measures $\mathbb{P}_{x_N}^N$, $N \geq 1$ is tight. Furthermore, we prove that every limit point of the sequence is concentrated on continuous trajectories. The proof of tightness is divided in several lemmas. We start with an estimate relating the sequence of operators \mathfrak{L}_N , $N \geq 1$ and the operator \mathfrak{L} . For $j \in S$ and $H \in C^2(\Sigma)$, recall the notation

$$(\Delta_j H)(x) := \sum_{k \in S} r(j, k) (\partial_{x_k} - \partial_{x_j})^2 H(x) .$$

Lemma 7.1. *If H belongs to \mathcal{D}_S ,*

$$\lim_{N \rightarrow \infty} \max_{x \in \Sigma_N} \left| (\mathfrak{L}_N H)(x) - (\mathfrak{L} H)(x) - \frac{1}{2} \sum_{j \in S} \{g_j(Nx_j) - m_j\} (\Delta_j H)(x) \right| = 0 .$$

In particular, there exists a finite constant $C_0 > 0$, which depends on H , such that

$$\sup_{N \geq 1} \max_{x \in \Sigma_N} \left| (\mathfrak{L}_N H)(x) \right| \leq C_0 .$$

Proof. Fix a function $H \in \mathcal{D}_S$. In view of (2.3), by Taylor expansion, for any $x \in \Sigma_N$,

$$(\mathfrak{L}_N H)(x) = N \sum_{j \in S} g_j(Nx_j) [\mathbf{v}_j \cdot \nabla H(x)] + \frac{1}{2} \sum_{j \in S} g_j(Nx_j) (\Delta_j H)(x) + R_N , \quad (7.1)$$

where $\lim_{N \rightarrow \infty} \max_{x \in \Sigma_N} |R_N| = 0$. Since $g_j(0) = 0$, we may introduce the indicator $\mathbf{1}\{x_j > 0\}$ in the first sum and write it as

$$N \sum_{j \in S} \left\{ \frac{g_j(Nx_j)}{m_j} - 1 \right\} \mathbf{1}\{x_j > 0\} m_j [\mathbf{v}_j \cdot \nabla H(x)] + N \sum_{j \in S} \mathbf{1}\{x_j > 0\} m_j [\mathbf{v}_j \cdot \nabla H(x)] . \quad (7.2)$$

Since H belongs to \mathcal{D}_S , $\mathbf{v}_j \cdot \nabla H(x) = 0$ for $x_j = 0$. We may therefore remove the indicator in the second sum. Since \mathbf{m} is an invariant measure for \mathbf{r} , $\sum_{j \in S} m_j \mathbf{v}_j = 0$. By these last observations, the second sum in (7.2) vanishes. The first term in (7.1) is thus equal to

$$\begin{aligned} & \sum_{j \in S} \left\{ Nx_j \left[\frac{g_j(Nx_j)}{m_j} - 1 \right] - b \right\} \mathbf{1}\{x_j > 0\} \frac{m_j}{x_j} [\mathbf{v}_j \cdot \nabla H(x)] \\ & + b \sum_{j \in S} \mathbf{1}\{x_j > 0\} \frac{m_j}{x_j} [\mathbf{v}_j \cdot \nabla H(x)] . \end{aligned} \quad (7.3)$$

The second term in (7.3) is $b(x) \cdot \nabla H(x)$, while the first term is uniformly small in view of (2.1) and because H belongs to \mathcal{D}_S . This completes the proof of the first assertion. The second assertion follows from the first one and from the fact that $\mathfrak{L}H$ is continuous on the compact Σ . \square

We start our route to the proof of Proposition 7.6 below by showing that tightness follows from an estimate, uniform over the initial position, of the evolution of the process in small time intervals.

Lemma 7.2. *Fix a sequence $x_N \in \Sigma_N$, $N \geq 1$. The sequence of probability measures $\mathbb{P}_{x_N}^N$ is tight if for any $\epsilon > 0$ and for any sequence $(N(k), t_{N(k)}, y_{N(k)})$ such that $N(k) \rightarrow \infty$, $y_{N(k)} \rightarrow y$ for some $y \in \Sigma$, $t_{N(k)} \rightarrow 0$,*

$$\lim_{k \rightarrow \infty} \mathbb{P}_{y_{N(k)}}^{N(k)} [\|X_{t_{N(k)}} - y_{N(k)}\| \geq \epsilon] = 0.$$

Proof. Fix a sequence $x_N \in \Sigma_N$, $N \geq 1$. By Aldous criterion, since Σ is a compact space, to prove that the sequence $\mathbb{P}_{x_N}^N$ is tight, it is enough to show that for every $T > 0$, $\epsilon > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{0 \leq t \leq \delta} \sup_{\tau} \mathbb{P}_{x_N}^N [\|X_{\tau+t} - X_{\tau}\| \geq \epsilon] = 0,$$

where the supremum is carried over all stopping times τ bounded by T . By the strong Markov property, to prove tightness it is therefore enough to show that for any $\epsilon > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \sup_{0 \leq t \leq \delta} \max_{x \in \Sigma_N} \mathbb{P}_x^N [\|X_t - x\| \geq \epsilon] = 0. \quad (7.4)$$

For each $\delta > 0$, there exists $t_N = t_N(\delta) \in [0, \delta]$ and $y_N = y_N(\delta) \in \Sigma_N$, $N \geq 1$, such that

$$\limsup_{N \rightarrow \infty} \sup_{0 \leq t \leq \delta} \max_{x \in \Sigma_N} \mathbb{P}_x^N [\|X_t - x\| \geq \epsilon] = \lim_{N \rightarrow \infty} \mathbb{P}_{y_N}^N [\|X_{t_N} - y_N\| \geq \epsilon].$$

On the right hand side the sequences t_N and y_N depend on δ , $t_N = t_N(\delta)$, $y_N = y_N(\delta)$. We may choose a further subsequence $\{N(k) : k \geq 1\}$ such that $\lim_k N(k) = \infty$, $t_{N(k)} \in [0, 1/k]$, and

$$\lim_{\delta \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{P}_{y_N}^N [\|X_{t_N} - y_N\| \geq \epsilon] = \lim_{k \rightarrow \infty} \mathbb{P}_{y_{N(k)}}^{N(k)} [\|X_{t_{N(k)}} - y_{N(k)}\| \geq \epsilon].$$

Since Σ is compact we may assume that $\lim_k y_{N(k)} = y \in \Sigma$. Therefore, if we are able to prove that for any $\epsilon > 0$, and any sequence $(N(k), t_{N(k)}, y_{N(k)})$ such that $N(k) \rightarrow \infty$, $y_{N(k)} \rightarrow y \in \Sigma$, $t_{N(k)} \rightarrow 0$,

$$\lim_{k \rightarrow \infty} \mathbb{P}_{y_{N(k)}}^{N(k)} [\|X_{t_{N(k)}} - y_{N(k)}\| \geq \epsilon] = 0,$$

(7.4) holds, and hence the sequence $\mathbb{P}_{x_N}^N$ is tight. This is the assertion of the lemma. \square

Denote by τ_δ , $\delta > 0$, the first time the process is at distance δ from its original position: $\tau_\delta = \inf\{t \geq 0 : \|X_t - X_0\| > \delta\}$. Let X^δ be the process X_t stopped at τ_δ :

$$X_t^\delta := X_{t \wedge \tau_\delta}.$$

Lemma 7.3. *Let $x_N \in \Sigma_N$ and $t_N > 0$, $N \geq 1$, be sequences such that $x_N \rightarrow x \in \Sigma$ and $t_N \rightarrow 0$. Let $B := \mathcal{B}(x)$, $A := \mathcal{A}(x)$ and let $\delta > 0$ be such that $\min_{j \in B} x(j) \geq 2\delta$. Then, for every $\epsilon > 0$ sufficiently small, and every function F in \mathcal{D}_A ,*

$$\lim_{N \rightarrow \infty} \mathbb{P}_{x_N}^N [|F(X_{t_N}^\delta) - F(x)| \geq \epsilon] = 0.$$

Proof. For N sufficiently large and $s \leq \tau_\delta$, $X_s^N \in \Lambda_B(\delta)$. Therefore, by Lemma 4.4, there exists a function $H \in \mathcal{D}_S$ such that

$$\mathbb{P}_{x_N}^N [|F(X_{t_N}^\delta) - F(x)| \geq \epsilon] = \mathbb{P}_{x_N}^N [|H(X_{t_N}^\delta) - H(x)| \geq \epsilon]. \quad (7.5)$$

Consider the $\mathbb{P}_{x_N}^N$ -martingale

$$M_t^N = H(X_t) - H(x) - \int_0^t (\mathfrak{L}_N H)(X_s) ds, \quad t \geq 0.$$

The probability appearing on the right hand side of (7.5) is bounded above by

$$\mathbb{P}_{x_N}^N \left[\left| \int_0^{t_N \wedge \tau_\delta} (\mathfrak{L}_N H)(X_s) ds \right| \geq \epsilon/2 \right] + \mathbb{P}_{x_N}^N [|M_{t_N \wedge \tau_\delta}^N| \geq \epsilon/2].$$

Therefore, since H belongs to \mathcal{D}_S , by the last assertion of Lemma 7.1, the time-integral appearing in the first term is absolutely bounded by $C_0 t_N$ for some finite constant C_0 which depends on δ and H . This proves that the first term in the previous displayed equation vanishes as $N \uparrow \infty$ because $t_N \downarrow 0$. By Tchebychef inequality, the second term is bounded by

$$\frac{4}{\epsilon^2} \mathbb{E}_{x_N}^N [(M_{t_N \wedge \tau_\delta}^N)^2] = \frac{4}{\epsilon^2} \mathbb{E}_{x_N}^N \left[\int_0^{t_N \wedge \tau_\delta} (\mathfrak{L}_N H^2 - 2H \mathfrak{L}_N H)(X_s) ds \right].$$

An elementary computation shows that $(\mathfrak{L}_N H^2 - 2H \mathfrak{L}_N H)(x)$ is absolutely bounded by a finite constant which depends on H , uniformly on Σ_N . This completes the proof of the lemma. \square

Corollary 7.4. *Under the assumptions of Lemma 7.3, for every $\epsilon > 0$,*

$$\lim_{N \rightarrow \infty} \mathbb{P}_{x_N}^N [\|X_{t_N}^\delta\|_A \geq \epsilon] = 0.$$

Proof. To estimate the first term on the right hand side, let $\Phi : [0, \infty) \rightarrow [0, \infty)$ be a smooth, non-decreasing function such that $\Phi(r) = 0$ for $0 \leq r \leq 1/2$, $\Phi(r) = r$ for $r \geq 1$. For $\epsilon > 0$, let $\Phi_\epsilon(r) = \epsilon \Phi(r/\epsilon)$, and recall the definition of the function J_A given in (4.3). Let

$$F_\epsilon(x) := c_1 \Phi_\epsilon(J_A(x)/c_1), \quad x \in \Sigma.$$

It is easy to check that F_ϵ belongs to \mathcal{D}_A and that $F_\epsilon(x) = J_A(x) \geq c_1 \|x\|_A \geq c_1 \epsilon$ if $\|x\|_A \geq \epsilon$. Therefore, for every $\epsilon > 0$,

$$\lim_{N \rightarrow \infty} \mathbb{P}_{x_N}^N [\|X_{t_N}^\delta\|_A \geq \epsilon] \leq \limsup_{N \rightarrow \infty} \mathbb{P}_{x_N}^N [F_\epsilon(X_{t_N}^\delta) \geq c_1 \epsilon].$$

The right hand side vanishes in view of Lemma 7.3 and because $F_\epsilon(x) = 0$. \square

Recall the linear map $\Upsilon : \Sigma \rightarrow \Sigma_B$ defined in (3.5). Denote by $\phi_j : \Sigma_B \rightarrow \mathbb{R}$ the coordinate map $\phi_j(x) = x_j$, $j \in B$. By Lemma 3.1, the function $\phi_j \circ \Upsilon$ belongs to \mathcal{D}_A for all $j \in B$ and so we may apply Lemma 7.3 for each $F = \phi_j \circ \Upsilon$.

Corollary 7.5. *Under the assumptions of Lemma 7.3, for any small enough $\epsilon > 0$,*

$$\lim_{N \rightarrow \infty} \mathbb{P}_{x_N}^N [\|X_{t_N}^\delta - x\|_B \geq \epsilon] = 0.$$

Proof. Since $\Upsilon(x) = x_B$, it is easy to verify that

$$\|X_{t_N}^\delta - x\|_B \leq \|\Upsilon(X_{t_N}^\delta) - \Upsilon(x)\|_B + C_0 \|X_{t_N}^\delta\|_A,$$

for some finite constant $C_0 > 0$. The assertion of the lemma follows therefore by applying Lemma 7.3 to each function $\phi_j \circ \Upsilon \in \mathcal{D}_A$, and by Corollary 7.4. \square

Proposition 7.6. *For any sequence $x_N \in \Sigma_N$, $N \geq 1$, the sequence of laws $\{\mathbb{P}_{x_N}^N : N \geq 1\}$ is tight. Moreover, every limit point of the sequence is concentrated on continuous trajectories.*

Proof. It is enough to prove that the conditions of Lemma 7.2 are in force. To keep notation simple, we show that the conditions are fulfilled for a sequence $t_N \rightarrow 0$, $x_N \rightarrow x \in \Sigma$, $x_N \in \Sigma_N$. Let $A = \mathcal{A}(x)$, $B = \mathcal{B}(x)$, and let $\delta > 0$ be such that $\min_{j \in B} x_j \geq 2\delta$.

Recall the definition of the stopped process X_t^δ introduced just before Lemma 7.3. To prove that

$$\lim_{N \rightarrow \infty} \mathbb{P}_{x_N}^N [\|X_{t_N} - x_N\| \geq \epsilon] = 0.$$

for $\epsilon < \delta$, it is enough to show that

$$\lim_{N \rightarrow \infty} \mathbb{P}_{x_N}^N [\|X_{t_N}^\delta - x\| \geq \epsilon] = 0.$$

Fix $0 < \epsilon < \delta$. Clearly, for N large enough,

$$\mathbb{P}_{x_N}^N [\|X_{t_N}^\delta - x\| \geq \epsilon] \leq \mathbb{P}_{x_N}^N [\|X_{t_N}^\delta\|_A \geq \epsilon/2] + \mathbb{P}_{x_N}^N [\|X_{t_N}^\delta - x\|_B \geq \epsilon/2].$$

To complete the proof of the first assertion of the proposition, it remains to apply Corollaries 7.4 and 7.5.

Any limit point of the sequence $\mathbb{P}_{x_N}^N$ is concentrated on continuous trajectories because for any $T > 0$, $\sup_{0 \leq t \leq T} \|X_t^N - X_{t-}^N\| \leq 2/N$. Moreover, it follows from the tightness of the sequence $\mathbb{P}_{x_N}^N$ that for every $\epsilon > 0$ and every sequence x_N ,

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{x_N}^N \left[\sup_{0 \leq t \leq \delta} \|X_t - X_0\| \geq \epsilon \right] = 0. \quad (7.6)$$

□

7.2. Characterization of limit points. We prove in this subsection that any limit point of a sequence $\mathbb{P}_{x_N}^N$ solves the martingale problem (2.8).

Proposition 7.7. *Let $x_N \in \Sigma_N$, $N \geq 1$, be a sequence converging to some $x \in \Sigma$, and denote by $\tilde{\mathbb{P}}$ a limit point of the sequence $\mathbb{P}_{x_N}^N$. Under $\tilde{\mathbb{P}}$, for any $H \in \mathcal{D}_S$,*

$$H(X_t) - H(X_0) - \int_0^t (\mathfrak{L}H)(X_s) ds$$

is a martingale.

The following replacement lemma is the key point in the proof of Proposition 7.7. It permits to replace the functions $g_\ell(NX_s(\ell))$ by the constants m_ℓ , $\ell \in S$.

Lemma 7.8. *For any $\ell \in S$,*

$$\lim_{N \rightarrow \infty} \max_{x \in \Sigma_N} \mathbb{E}_x^N \left[\left(N \int_0^{1/N} \{g_\ell(NX_s(\ell)) - m_\ell\} ds \right)^2 \right] = 0. \quad (7.7)$$

Proof. Let x_N be a sequence such that

$$\begin{aligned} & \lim_{N \rightarrow \infty} \max_{x \in \Sigma_N} \mathbb{E}_x^N \left[\left(N \int_0^{1/N} \{g_\ell(NX_s(\ell)) - m_\ell\} ds \right)^2 \right] \\ &= \lim_{N \rightarrow \infty} \mathbb{E}_{x_N}^N \left[\left(N \int_0^{1/N} \{g_\ell(NX_s(\ell)) - m_\ell\} ds \right)^2 \right]. \end{aligned}$$

Assume without loss of generality that x_N converges to some $x \in \Sigma$. Fix $j \in S$, and suppose first that $x(j) > 0$. In this case, since $\lim_n g_j(n) = m_j$, the assertion of the lemma for $\ell = j$ follows from (7.6). If $x(j) = 0$, there exists $k \neq j$ such that $x(k) > 0$. By the previous observation, (7.7) holds with k in place of ℓ .

For $j, k \in S$, consider the function $u : S \rightarrow \mathbb{R}$ defined by

$$u(j) = 1, \quad u(k) = 0 \quad \text{and} \quad (\mathcal{L}u)(i) = 0 \text{ for } i \neq j, k,$$

let $F(x) = u \cdot x$, and let $(M_t)_{t \geq 0}$ be the Dynkin's martingale associated to F :

$$M_t := F(X_t) - F(X_0) - N \int_0^t \sum_{i \in S} g_i(NX_s(i)) (\mathcal{L}f)(i) ds, \quad t \geq 0.$$

On the one hand, an elementary computation shows that

$$\mathbb{E}_{x_N}^N [M_t^2] = \mathbb{E}_{x_N}^N \left[\int_0^t \sum_{i, \ell \in S} g_i(NX_s(i)) r(i, \ell) [u(\ell) - u(i)]^2 ds \right],$$

so that $\lim_N \mathbb{E}_{x_N}^N [M_{1/N}^2] = 0$. On the other hand, by definition of F and by (7.6),

$$\lim_{N \rightarrow \infty} \mathbb{E}_{x_N}^N [\{F(X_{1/N}) - F(X_0)\}^2] = 0.$$

Therefore,

$$\lim_{N \rightarrow \infty} \mathbb{E}_{x_N}^N \left[\left(N \int_0^{1/N} \sum_{i \in S} g_i(NX_s(i)) (\mathcal{L}u)(i) ds \right)^2 \right] = 0.$$

As $\sum_{i \in S} m_i (\mathcal{L}u)(i) = 0$, we may substitute in the previous equation $g_i(NX_s(i))$ by $g_i(NX_s(i)) - m_i$. Since $(\mathcal{L}u)(i) = 0$ for $i \neq j, k$, since $(\mathcal{L}u)(j) \neq 0$, and since (7.7) holds for $\ell = k$,

$$\lim_{N \rightarrow \infty} \mathbb{E}_{x_N}^N \left[\left(N \int_0^{1/N} \{g_j(NX_s(j)) - m_j\} ds \right)^2 \right] = 0,$$

which completes the proof of the lemma. \square

Corollary 7.9. *For any $t > 0$, $j \in S$, and any continuous function $H : \Sigma \rightarrow \mathbb{R}$,*

$$\lim_{N \rightarrow \infty} \max_{x \in \Sigma_N} \mathbb{E}_x^N \left[\left| \int_0^t \{g_j(NX_s(j)) - m_j\} H(X_s) ds \right| \right] = 0.$$

Proof. Fix a sequence $x_N \in \Sigma_N$, $N \geq 1$ some $t > 0$, $j \in S$, and a continuous function $H : \Sigma \rightarrow \mathbb{R}$. Clearly,

$$\begin{aligned} & \mathbb{E}_{x_N}^N \left[\left| \int_0^t \{g_j(NX_s(j)) - m_j\} H(X_s) ds \right| \right] \\ & \leq \sum_{k=0}^{[tN]} \mathbb{E}_{x_N}^N \left[\left| \int_{k/N}^{(k+1)/N} \{g_j(NX_s(j)) - m_j\} H(X_s) ds \right| \right] + O\left(\frac{1}{N}\right), \end{aligned}$$

where $[a]$ stands for the integer part of $a \in \mathbb{R}$. By the Markov property, the first term on the right hand side is bounded by

$$[tN] \max_{x \in \Sigma_N} \mathbb{E}_x^N \left[\left| \int_0^{1/N} \{g_j(NX_s(j)) - m_j\} H(X_s) ds \right| \right].$$

Since H is a continuous function, for every $\delta > 0$, there exists $\epsilon > 0$ such that the previous expression is bounded by

$$\begin{aligned} & \delta + C_0 \max_{x \in \Sigma_N} \mathbb{P}_x^N \left[\sup_{0 \leq s \leq 1/N} \|X_s - x\| \geq \epsilon \right] \\ & + [tN] \max_{x \in \Sigma} |H(x)| \max_{x \in \Sigma_N} \mathbb{E}_x^N \left[\left| \int_0^{1/N} \{g_j(NX_s(j)) - m_j\} ds \right| \right] \end{aligned}$$

for some finite constant C_0 which depends on H and t . By (7.6), the second term vanishes as $N \uparrow \infty$. The third one vanishes by Lemma 7.8. \square

Proof of Proposition 7.7. Fix a function H in \mathcal{D}_S , $n \geq 1$, a continuous function $G : \Sigma^n \rightarrow \mathbb{R}$, and $0 \leq s_1 \leq \dots \leq s_n \leq t_1 < t_2$. Define

$$G(X) := G(X_{s_1}, \dots, X_{s_n}) \quad \text{and} \quad \Psi_{t_1, t_2} := H(X_{t_2}) - H(X_{t_1}) - \int_{t_1}^{t_2} (\mathfrak{L}H)(X_r) dr.$$

Fix a sequence $x_N \in \Sigma_N$, $N \geq 1$ and let $\tilde{\mathbb{P}}$ be a limit point of the sequence $\mathbb{P}_{x_N}^N$. Assume, without loss of generality, that $\mathbb{P}_{x_N}^N$ converges to $\tilde{\mathbb{P}}$ in the Skorohod topology. For each $N \geq 1$,

$$\mathbb{E}_{x_N}^N \left[G(X) \left\{ H(X_{t_2}) - H(X_{t_1}) - \int_{t_1}^{t_2} (\mathfrak{L}_N H)(X_r) dr \right\} \right] = 0.$$

By Lemma 7.1, this left hand side is equal to

$$\mathbb{E}_{x_N}^N \left[G(X) \Psi_{t_1, t_2} \right] + \frac{1}{2} \sum_{j \in S} \mathbb{E}_{x_N}^N \left[G(X) \int_{t_1}^{t_2} \{ g_j(NX_s(j)) - m_j \} (\Delta_j H)(X_s) ds \right]$$

plus a remainder which vanishes as $N \uparrow \infty$. By the Markov property and by Corollary 7.9, the second term vanishes as $N \uparrow \infty$. Since $\mathbb{P}_{x_N}^N$ converges in the Skorohod topology to $\tilde{\mathbb{P}}$ and since the measure $\tilde{\mathbb{P}}$ is concentrated on continuous paths, the first term converges to

$$\tilde{\mathbb{E}} \left[G(X) \left\{ H(X_{t_2}) - H(X_{t_1}) - \int_{t_1}^{t_2} (\mathfrak{L}H)(X_r) dr \right\} \right].$$

Putting together the previous estimates we conclude that this latter expectation vanishes. This completes the proof of the proposition. \square

For each $x \in \Sigma$, consider a sequence $x_N \in \Sigma_N$, $N \geq 1$ converging to x and a limit point $\tilde{\mathbb{P}}_x$ for the sequence $\mathbb{P}_{x_N}^N$, $N \geq 1$. Since $\tilde{\mathbb{P}}_x$ is concentrated on $C(\mathbb{R}_+, \Sigma)$ the restriction of $\tilde{\mathbb{P}}_x$ to this space turns out to be a probability measure starting at x . By Proposition 7.7, such restriction is a solution of the \mathfrak{L} -martingale problem starting at x . We have thus proved the existence of solutions for the \mathfrak{L} -martingale problem and the proof of Theorem 2.2 is concluded. On the other hand, Theorem 2.6 is an immediate consequence of Proposition 7.7 and of the uniqueness of the \mathfrak{L} -martingale problem established in the last section.

7.3. Additional Properties. In this subsection we prove some additional properties of the solution $\{\mathbb{P}_x : x \in \Sigma\}$. We start showing Feller continuity.

Proposition 7.10. *Let $(x_n)_{n \geq 1}$ be a sequence in Σ converging to some $x \in \Sigma$. Then $\mathbb{P}_{x_n} \rightarrow \mathbb{P}_x$ in the sense of weak convergence of measures on $C(\mathbb{R}_+, \Sigma)$.*

Proof. For each $z \in \Sigma$, let $\tilde{\mathbb{P}}_z$ stand for the probability measure on $D(\mathbb{R}_+, \Sigma)$ induced by \mathbb{P}_z and the inclusion of $C(\mathbb{R}_+, \Sigma)$ into $D(\mathbb{R}_+, \Sigma)$, and denote by $\tilde{\mathbb{E}}_z$ the respective expectation.

Fix a bounded, continuous function $\Gamma : D(\mathbb{R}_+, \Sigma) \rightarrow \mathbb{R}$. By Theorem 2.6, there exists a strictly increasing sequence $N(n) \in \mathbb{N}$, $n \geq 1$, and a sequence $(z_n)_{n \geq 1}$ so that $z_n \in \Sigma_{N(n)}$,

$$\|z_n - x_n\| < \frac{1}{n} \quad \text{and} \quad \left| \mathbb{E}_{z_n}^{N(n)}[\Gamma] - \tilde{\mathbb{E}}_{x_n}[\Gamma] \right| < \frac{1}{n}, \quad (7.8)$$

for all $n \geq 1$. In particular, $z_n \rightarrow x$. By Theorem 2.6,

$$\mathbb{E}_{z_n}^{N(n)}[\Gamma] \rightarrow \tilde{\mathbb{E}}_x[\Gamma], \quad \text{as } n \uparrow \infty.$$

From the previous convergence and from the second assertion in (7.8), it follows that $\tilde{\mathbb{E}}_{x_n}[\Gamma] \rightarrow \tilde{\mathbb{E}}_x[\Gamma]$, as $n \uparrow \infty$. We have thus shown that $\tilde{\mathbb{P}}_{x_n} \rightarrow \tilde{\mathbb{P}}_x$ in the sense of

weak convergence of measures in $D(\mathbb{R}_+, \Sigma)$. Since every $\tilde{\mathbb{P}}_z$, $z \in \Sigma$, is concentrated on $C(\mathbb{R}_+, \Sigma)$, this implies the desired result. \square

Next result asserts that $\{\mathbb{P}_x : x \in \Sigma\}$ satisfies the strong Markov property.

Proposition 7.11. *Fix $x \in \Sigma$. Let τ be a finite stopping time and $\{\mathbb{P}_\omega^\tau\}$ be a r.c.p.d. of \mathbb{P}_x given \mathcal{F}_τ . Then, there exists a \mathbb{P}_x -null set $\mathcal{N} \in \mathcal{F}_\tau$, such that*

$$\mathbb{P}_\omega^\tau \circ \theta_{\tau(\omega)}^{-1} = \mathbb{P}_{X_\tau(\omega)}, \quad \omega \in \mathcal{N}^c,$$

where we recall $(\theta_t)_{t \geq 0}$ is the semigroup of time translations.

Proof. By Theorems 2.3 and 2.5, \mathbb{P}_x is an absorbing solution of the \mathcal{L} -martingale problem. The same argument used in Lemma 6.3 shows that there exists a \mathbb{P}_x -null set $\mathcal{N} \in \mathcal{F}_\tau$ such that for all $\omega \in \mathcal{N}^c$ the probability $\mathbb{P}_\omega^\tau \circ \theta_{\tau(\omega)}^{-1}$ is an absorbing solution of the \mathcal{L} -martingale problem starting at $X_\tau(\omega)$. By the uniqueness result in Proposition 6.1 and by Theorem 2.5, we conclude that $\mathbb{P}_\omega^\tau \circ \theta_{\tau(\omega)}^{-1} = \mathbb{P}_{X_\tau(\omega)}$ for all $\omega \in \mathcal{N}^c$, which is the content of the proposition. \square

To conclude this section we give an estimate for the expected value of the absorbing time σ_1 uniformly on the starting point $x \in \Sigma$.

Proposition 7.12. *Let $z \in \Sigma$ be such that $z \neq e_j$, $j \in S$. For any $q > b$,*

$$\mathbb{E}_z[\sigma_1] \leq \frac{|B|^{(q-1) \vee 1}}{(q+1)(q-b)d(B)},$$

where $B = \mathcal{B}(z)$ and $d(B) := \min_{j \in B} \langle (-\mathcal{S})e_j, e_j \rangle_m$. In particular, $\mathbb{P}_x[\sigma_1 < \infty] = 1$.

Proof. Fix $q > b$, $z \in \Sigma$, and let $B = \mathcal{B}(z)$. For $j \in B$ and $\epsilon > 0$, there exists a function $F_\epsilon \in D_0(\Sigma)$ such that

$$F_\epsilon(x) = x_j^{q+1}, \quad x \in \Sigma_{B,0} \cap \Lambda_B(\epsilon/2).$$

Recall the definition of the stopping time $h_B(\epsilon)$ given in (5.9). By Theorem 2.5,

$$F_\epsilon(X_{t \wedge h_B(\epsilon)}) - \int_0^{t \wedge h_B(\epsilon)} (\mathcal{L}F_\epsilon)(X_s) ds$$

is a \mathbb{P}_z -martingale, so that

$$\mathbb{E}_z \left[\int_0^{t \wedge h_B(\epsilon)} (\mathcal{L}F_\epsilon)(X_s) ds \right] = \mathbb{E}_z[F_\epsilon(X_{t \wedge h_B(\epsilon)}) - F(z)] \leq 1. \quad (7.9)$$

By definition of F_ϵ , for all $x \in \Sigma_{B,0} \cap \Lambda_B(\epsilon)$ we have

$$\mathcal{L}F_\epsilon(x) = b(q+1) \sum_{k \in B} \frac{x_j^q}{x_k} m_k \mathcal{L}^B e_j(k) + q(q+1) x_j^{q-1} \langle (-\mathcal{S}^B)e_j, e_j \rangle_m, \quad (7.10)$$

Since $\mathcal{L}^B e_j(k) \geq 0$ for $k \neq j$ and $m_j \mathcal{L}^B e_j(j) = -D_B(e_j, e_j)$ then the expression in (7.10) is bounded below by

$$(q-b)(q+1) x_j^{q-1} D_B(e_j, e_j).$$

By using this bound in (7.9), definition of $h_B(\epsilon)$ and the absorbing property we get

$$\mathbb{E}_z \left[\int_0^{t \wedge h_B(\epsilon)} X_s(j)^{q-1} ds \right] \leq \frac{1}{(q-b)(q+1)d(B)}$$

Averaging over $j \in B$ in the above inequality and by using that

$$\frac{1}{|B|} \sum_{j \in B} x_j^{q-1} \geq \frac{1}{|B|^{(q-1) \vee 1}}$$

we obtain

$$\mathbb{E}_z[t \wedge h_B(\epsilon)] \leq \frac{|B|^{(q-1) \vee 1}}{(q-b)(q+1)d(B)}.$$

It remains to let $t \uparrow \infty$ to complete the proof of the lemma. \square

8. PROOF OF LEMMA 4.1

In this section we prove Lemma 4.1, which has been used in Lemmas 4.3 and 4.4 for the construction of suitable functions and in Corollary 7.4 for the proof of tightness.

Recall that A is a proper nonempty subset of S . For each $j \in A$ let \mathbf{w}_j represent the canonical projection of the vector \mathbf{v}_j on \mathbb{R}^A . Also let $\{\mathbf{e}_k : k \in A\}$ represent here the canonical basis of \mathbb{R}^A . We start observing the following relation between these two sets of vectors.

Lemma 8.1. *Let D and D' be two different subsets of A and suppose that*

$$\sum_{j \in D} \alpha_j \mathbf{w}_j + \sum_{k \in A \setminus D} \alpha_k \mathbf{e}_k = \sum_{j \in D'} \beta_j \mathbf{w}_j + \sum_{k \in A \setminus D'} \beta_k \mathbf{e}_k, \quad (8.1)$$

for some $\alpha_j, \beta_j \in \mathbb{R}$ such that $\beta_j \alpha_j \geq 0$ for all $j \in A$. Then $\alpha_j = \beta_j$ for all $j \in A$ and $\alpha_k = \beta_k = 0, \forall k \in D \Delta D'$.

Proof. Without loss of generality we may suppose that $\alpha_j \geq 0$ and $\beta_j \geq 0$, for all $j \in A$. Otherwise, we may interchange the respective terms in equation (8.1). Fix an arbitrary $j_0 \in D \setminus D'$ and consider the vector $u \in \mathbb{R}^S$ defined by $u(j_0) = 1$ and

$$\begin{cases} u(k) = 0, & \text{for } k \in S \setminus D; \\ \mathcal{L}u(j) = 0, & \text{for } j \in D \setminus \{j_0\}. \end{cases}$$

Since $u \equiv 0$ on $S \setminus D$ then the inner product of the canonical projection of u on \mathbb{R}^A and the expression in the left hand side of (8.1) equals

$$\left(\sum_{j \in D} \alpha_j \mathbf{v}_j \right) \cdot u = \sum_{j \in D} \alpha_j \mathcal{L}u(j). \quad (8.2)$$

Since $\mathcal{L}u \equiv 0$ on $D \setminus \{j_0\}$, the last expression reduces to $\alpha_{j_0} \mathcal{L}u(j_0)$. On the other hand, the inner product of the canonical projection of u on \mathbb{R}^A and the expression in the right hand side of (8.1) is equal to

$$\sum_{j \in D'} \beta_j \mathcal{L}u(j) + \sum_{k \in A \setminus D'} \beta_k u(k). \quad (8.3)$$

Since $u \in [0, 1]^S$ and $u \equiv 0$ on $D' \setminus D$ then $\mathcal{L}u(j) \geq 0$ for all $j \in D' \setminus D$ and so, the first term in (8.3) is positive. Therefore, (8.3) is bounded below by

$$\sum_{k \in A \setminus D'} \beta_k u(k) = \sum_{k \in D \setminus D'} \beta_k u(k).$$

From (8.2) and the last estimate we conclude that

$$\alpha_{j_0} \mathcal{L}u(j_0) \geq \sum_{k \in D \setminus D'} \beta_k u(k) \geq \beta_{j_0}. \quad (8.4)$$

Since \mathbf{r} is irreducible and $S \setminus D$ is not empty (A is a proper subset of S) then $\mathcal{L}u(j_0)$ is strictly negative. By using this observation in inequality (8.4) we get $\alpha_{j_0} = \beta_{j_0} = 0$. By interchanging D and D' in the argument we finally conclude that

$$\alpha_k = \beta_k = 0, \quad \forall k \in D \Delta D'. \quad (8.5)$$

By inserting (8.5) in (8.1) we get

$$\sum_{j \in \tilde{D}} \alpha_j \mathbf{w}_j + \sum_{k \in \tilde{C}} \alpha_k \mathbf{e}_k = \sum_{j \in \tilde{D}} \beta_j \mathbf{w}_j + \sum_{k \in \tilde{C}} \beta_k \mathbf{e}_k, \quad (8.6)$$

where $\tilde{D} = D \cap D'$ and $\tilde{C} = A \setminus (D \cup D')$. Fix an arbitrary $j_1 \in \tilde{D}$ and consider now the vector $v \in \mathbb{R}^S$ defined by $v(j_1) = 1$ and

$$\begin{cases} v(j) = 0, & \text{for } j \in S \setminus \tilde{D}; \\ \mathcal{L}v(j) = 0, & \text{for } j \in \tilde{D} \setminus \{j_1\}. \end{cases}$$

Similarly to the computation we performed for equation (8.1), we take the inner product of the canonical projection of v on \mathbb{R}^A and each term in equation (8.6) to get

$$\alpha_{j_1} \mathcal{L}v(j_1) = \beta_{j_1} \mathcal{L}v(j_1). \quad (8.7)$$

Since $S \setminus \tilde{D}$ is nonempty and \mathbf{r} is irreducible then $\mathcal{L}v(j_1) < 0$ implying that $\alpha_{j_1} = \beta_{j_1}$. We have thus proved that $\alpha_j = \beta_j$ for all $j \in \tilde{D}$. Therefore, from (8.6) we conclude that actually $\alpha_j = \beta_j$ for all $j \in \tilde{D} \cup \tilde{C}$. \square

Corollary 8.2. *For any $D \subseteq A$, the set of vectors*

$$\{\mathbf{w}_j : j \in D\} \cup \{\mathbf{e}_k : k \in A \setminus D\}$$

is a basis of \mathbb{R}^A .

Proof. Suppose that

$$\sum_{j \in D} \alpha_j \mathbf{w}_j + \sum_{k \in A \setminus D} \alpha_k \mathbf{e}_k = 0.$$

By applying the previous lemma with $\beta_j = 0, j \in A$ and $D' = A \setminus D$ we get $\alpha_j = 0$ for all $j \in A$ proving the desired result. \square

Corollary 8.3. *For every $v \in \mathbb{R}^A$ there exists $D \subseteq A$ such that*

$$v = \sum_{j \in D} \alpha_j \mathbf{w}_j + \sum_{k \in A \setminus D} \alpha_k \mathbf{e}_k$$

with $\alpha_j \geq 0$, for all $j \in A$.

Proof. Denote by \mathcal{W} the set of vectors in \mathbb{R}^A for which all the coordinates with respect to the basis

$$\{\mathbf{w}_j : j \in D\} \cup \{\mathbf{e}_k : k \in A \setminus D\} \quad (8.8)$$

are non zero, for every $D \subseteq A$. Fix some $x \in \mathcal{W}$. For each $D \subseteq A$, denote by $\sigma_D \in \{-1, +1\}^S$ the vector

$$\sigma_D(j) := \begin{cases} +1, & \text{if } \alpha_j > 0, \\ -1, & \text{if } \alpha_j < 0, \end{cases}$$

where $\alpha_j, j \in A$ are the coordinates of x with respect to the basis (8.8). By Lemma 8.1, $\sigma_D \neq \sigma_{D'}$ for $D \neq D'$. Since $\{-1, +1\}^S$ and the powerset of A have the same cardinality then there must exist some $D_0 \subseteq A$ such that $\sigma_{D_0} \equiv +1$. This shows the assertion for every vector in \mathcal{W} . Since \mathcal{W} is dense in \mathbb{R}^A and the set of vectors satisfying the assertion is closed in \mathbb{R}^A then the proof is complete. \square

We will also need the following observation

Lemma 8.4. *Fix some $D \subseteq A$, $j_0 \in D$ and write*

$$e_{j_0} = \sum_{j \in D} \alpha_j w_j + \sum_{k \in A \setminus D} \alpha_k e_k. \quad (8.9)$$

We have $\alpha_k \geq 0$, $k \in A \setminus D$.

Proof. Fix an arbitrary $k \in A \setminus D$ and consider the vector $v \in \mathbb{R}^S$ defined by $v(k) = 1$, $v \equiv 0$ on $(D \cap \{k\})^c$ and $\mathcal{L}v \equiv 0$ on D . Taking the inner product of the projection of v on \mathbb{R}^A and each term in equation (8.9) we get $v(j_0) = \alpha_k$. Since $v \in [0, 1]^S$, we are done. \square

For each subset D of A , let \mathfrak{C}_D be the closed cone generated by the vectors in (8.8):

$$\mathfrak{C}_D := \left\{ \sum_{j \in D} \alpha_j w_j + \sum_{k \in A \setminus D} \alpha_k e_k : \alpha_j \geq 0, j \in A \right\}.$$

By Corollary 8.2, each cone \mathfrak{C}_D is $|A|$ -dimensional and by Corollary 8.3 we have

$$\bigcup_{D: D \subseteq A} \mathfrak{C}_D = \mathbb{R}^A. \quad (8.10)$$

As an immediate consequence of Lemma 8.1 we have, for any two subsets D, D' of A , that

$$\mathfrak{C}_D \cap \mathfrak{C}_{D'} = \left\{ \sum_{\tilde{D}} \alpha_j w_j + \sum_{k \in \tilde{C}} \alpha_j e_k : \alpha_j \geq 0, j \in \tilde{D} \cup \tilde{C} \right\} \quad (8.11)$$

where $\tilde{D} = D \cap D'$ and $\tilde{C} = A \setminus (D \cup D')$. Note that \mathfrak{C}_\emptyset corresponds to the positive quadrant, $\mathfrak{C}_\emptyset = \mathbb{R}_+^A$. We also note that

$$\mathfrak{C}_A \subseteq \left\{ x \in \mathbb{R}^A : \sum_{j \in A} x_j \leq 0 \right\}. \quad (8.12)$$

Indeed, if $x = \sum_{j \in A} \alpha_j w_j$ with $\alpha_j \geq 0$ for all $j \in A$ and $\mathbf{1}_A \in \mathbb{R}^S$ is the indicator of $A \subsetneq S$ then

$$\sum_{j \in A} x_j = \left(\sum_{j \in A} \alpha_j v_j \right) \cdot \mathbf{1}_A = \sum_{j \in A} \alpha_j \mathcal{L} \mathbf{1}_A(j) \leq 0$$

because $\mathcal{L} \mathbf{1}_A(j) \leq 0$ for all $j \in A$.

We finally need the following observation about the position of the cones.

Lemma 8.5. *We have*

$$\{x \in \mathbb{R}^A : x_k \leq 0\} \subseteq \bigcup_{D: k \in D} \mathfrak{C}_D.$$

for any $k \in A$.

Proof. Let $x \in \mathbb{R}^A$ be such that $x_k < 0$. By (8.10), $x \in \mathfrak{C}_D$ for some $D \subseteq A$. Since among the vectors $\{w_j : j \in B\}, \{e_j : j \in B\}$ only the vector w_k has its k -th coordinate negative, D must contain k . Since the cones are closed, the assertion follows from this observation. \square

We may now conclude the proof of Lemma 4.1. For each $\epsilon > 0$ denote $Q_\epsilon = [-\epsilon, \infty)^A$. Let $G : \mathbb{R}_+^A \rightarrow \mathbb{R}$ be the function defined by

$$G(x) = \sum_{j \in A} x_j .$$

The idea of the proof is to extend G to Q_ϵ in a linear way for boundary conditions in Definition 2.1 to be fulfilled in a neighborhood of the boundary of Q_ϵ . We first extend G to \mathbb{R}^A as follows. Fix $x \in \mathbb{R}^A$. According to (8.10), $x \in \mathfrak{C}_D$ for some $D \subseteq A$. We then define

$$F(x) = G\left(\sum_{j \in A \setminus D} \alpha_k e_k\right) .$$

where $\alpha_k \geq 0$, $k \in A$ are the coordinates of x in the basis (8.8). Observation (8.11) assures that F is well defined. Since $\mathfrak{C}_\emptyset = \mathbb{R}_+^A$ then F and G coincide on \mathbb{R}_+^A . Moreover, F is constant along the vector w_j in the cone \mathfrak{C}_D if D contains j . In particular, by Lemma 8.5,

$$\partial_{w_j} F(x) = 0 \quad \text{for } x \text{ such that } x_j < 0 , \quad (8.13)$$

where ∂_{w_j} stands for the directional derivative along w_j . It is also clear from the definition of the function F that in the interior of the cone \mathfrak{C}_D , denoted hereafter by $\mathring{\mathfrak{C}}_D$, F increases in the e_k direction for $k \notin D$. Actually, $(\partial_{x_k} F)(x) = 1$ for $x \in \mathring{\mathfrak{C}}_D$, $k \notin D$. Moreover, in view of Lemma 8.4, in \mathfrak{C}_D , $(\partial_{x_k} F)(x) \geq 0$ for $k \in D$. Therefore, in the interior of the cone \mathfrak{C}_D

$$(\partial_{x_k} F)(x) = 1 \quad \text{for } k \notin D , \quad \text{and} \quad (\partial_{x_j} F)(x) \geq 0 \quad \text{for } j \in D . \quad (8.14)$$

The function F is clearly not C^2 because its partial derivatives are not continuous. To remedy, we convolve it with a smooth mollifier. Let $\varphi : \mathbb{R}^A \rightarrow \mathbb{R}_+$ be a mollifier: φ is a smooth function whose support is contained in $[-|A|^{-1/2}, |A|^{-1/2}]^A$, and $\int_{\mathbb{R}^A} \varphi(x) dx = 1$. For $\delta > 0$, let $\varphi_\delta(x) = \delta^{-|A|} \varphi(x/\delta)$. Fix $\delta > 0$, and denote by $F_\delta : \mathbb{R}^A \rightarrow \mathbb{R}$ the function obtained by taking the convolution of F with φ_δ . Clearly, the function F_δ is smooth. Since the boundaries of the cones have null Lebesgue measure, by (8.13) and (8.14),

$$(w_j \cdot \nabla F_\delta)(x) = 0 \quad \text{for } x_j \leq -\delta , \quad \sum_{k \in A} (\partial_{x_k} F_\delta)(x) \geq 1 \quad \text{if } d(x, \mathfrak{C}_A) \geq \delta , \quad (8.15)$$

where $d(x, \mathfrak{C}_A)$ represents the distance from x to \mathfrak{C}_A .

Fix $\epsilon \geq \max\{2, |A|^{1/2}\}\delta$, and let $\mathfrak{S}_a^+ = \{x \in \mathbb{R}^A : \sum_{j \in A} x_j \geq a\}$, $\mathfrak{S}_b^- = \{x \in \mathbb{R}^A : \sum_{j \in A} x_j \leq b\}$. Since $d(\mathfrak{S}_\epsilon^+, \mathfrak{S}_0^-) = \epsilon/|A|^{1/2} \geq \delta$ and since, by (8.12), $\mathfrak{C}_A \subseteq \mathfrak{S}_0^-$, the second property in (8.15) holds in \mathfrak{S}_ϵ^+ .

Recall that $Q_\epsilon = [-\epsilon, \infty)^A$, and let $A = \max\{F_\delta(x) : x \in Q_\epsilon \cap \mathfrak{S}_\epsilon^-\}$. The constant A is finite because F_δ is a continuous function and $Q_\epsilon \cap \mathfrak{S}_\epsilon^-$ is a compact set. Fix $a > A$, and denote by $\mathfrak{M} \subset Q_\epsilon$ the a -level set of F_δ in Q_ϵ : $\mathfrak{M} = \{x \in Q_\epsilon : F_\delta(x) = a\}$. By the choice of a , \mathfrak{M} is contained in the interior of \mathfrak{S}_ϵ^+ . Let \mathfrak{S}_+ be the intersection of the radius-one sphere with \mathbb{R}_+^A : $\mathfrak{S}_+ = \{x \in \mathbb{R}_+^A : \|x\| = 1\}$. Let ϵ be the vector $(-\epsilon, \dots, -\epsilon)$. For each $x \in \mathfrak{S}_+$, there exists a unique $r > 0$ such that $\epsilon + rx \in \mathfrak{M}$, that is, such that $F_\delta(\epsilon + rx) = 1$. Existence follows from the continuity of F_δ and from the fact that $F_\delta(\epsilon) = 0$, $\lim_{r \rightarrow \infty} F_\delta(\epsilon + rx) = \infty$. The point r is unique because $F_\delta(\epsilon + rx)$ is strictly increasing in r in the set \mathfrak{S}_ϵ^+ in view of the second property in (8.15), and because the level set \mathfrak{M} is contained in the interior of \mathfrak{S}_ϵ^+ by definition of a .

We are finally in a position to define the function I_A . Let $J_A : \Sigma \rightarrow \mathbb{R}_+$ be given by $J_A(x) = 0$ if $x_A = 0$ and, otherwise,

$$J_A(x) = s ,$$

where s is the unique $r > 0$ such that $\epsilon + x_A/r \in \mathfrak{M}$. Note that the canonical projection of the level set $\{x \in \Sigma : J_A(x) = 1\}$ on \mathbb{R}^A corresponds to the set $-\epsilon + \mathfrak{M}$. It is clear from the definition of J_A that there exists finite constants $0 < c_1 < C_1$ such that

$$c_1 \|x\|_A \leq J_A(x) \leq C_1 \|x\|_A .$$

We then set $I_A(x) = J_A(x)^2$, $x \in \Sigma$. It is not difficult to check that $I_A \in C^2(\Sigma)$ because the manifold \mathfrak{M} is smooth. By the first property in (8.15), for each $x \in \Sigma$, $j \in A$ such that $x_j = 0$, there exists a neighborhood of x in which the function $J_A(x)$ is constant along the v_j -direction. Therefore I_A belongs to \mathcal{D}_A . This completes the proof of Lemma 4.1.

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